

Generalized $\mathcal{N} = 2$ Topological Amplitudes and Holomorphic Anomaly Equation

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Abstract

In arXiv:0905.3629 we described a new class of $\mathcal{N} = 2$ topological amplitudes that depends both on vector and hypermultiplet moduli. Here we find that this class is actually a particular case of much more general topological amplitudes which appear at higher loops in heterotic string theory compactified on $K3 \times T^2$. We analyze their effective field theory interpretation and derive particular (first order) differential equations as a consequence of supersymmetry Ward identities and the 1/2-BPS nature of the corresponding effective action terms. In string theory the latter get modified due to anomalous world-sheet boundary contributions, generalizing in a non-trivial way the familiar holomorphic and harmonicity anomalies studied in the past. We prove by direct computation that the subclass of topological amplitudes studied in arXiv:0905.3629 forms a closed set under these anomaly equations and that these equations are integrable.

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1 Introduction

Over the last couple of decades, considerable effort has been put into the study of 1/2-BPS effective couplings in four-dimensional extended supersymmetric theories. Such couplings depend only on half of the superspace, generalizing the well-known chiral F-terms of $\mathcal{N} = 1$ supersymmetry. They usually enjoy particular non-renormalization theorems which is the reason why they are easier to study from a field theoretic point of view. Moreover, these couplings play a predominant role in many very interesting applications, ranging from string phenomenology to entropy corrections of supersymmetric black holes. It is therefore of considerable interest to classify different kinds of 1/2-BPS couplings and study their properties in as much generality as possible.

In the string theory effective action, 1/2-BPS couplings are expected to be captured by

topological amplitudes, i.e. amplitudes which only depend on the zero-mode structure of the internal manifold [1, 2, 3]. The best studied example in this respect is the celebrated series $F_g \mathcal{W}^{2g}$ where \mathcal{W} is the chiral $\mathcal{N} = 2$ Weyl superfield of the supergravity multiplet. The coefficients F_g are computed as g loop amplitudes in type II string theory compactified on a Calabi-Yau threefold [2, 3] and are functions of the vector multiplet moduli in the Coulomb phase of the theory. In particular, their independence of $\mathcal{N} = 2$ hypermultiplet moduli (and therefore also of the type II dilaton) implies a non-renormalization theorem that F_g is g -loop exact. Moreover, classically their BPS nature dictates that the F_g are holomorphic functions. This property is broken at the quantum level, however, in a controlled manner captured by a first order differential equation, termed the *holomorphic anomaly equation* [3]. The latter takes the form of a recursion relation in g which in fact allows in some cases to find explicit expressions for F_g up to considerably high genus [4, 5, 6].

The heterotic versions of the F_g 's have been considered in [7]. They are semi-topological expressions (i.e. topological only in the supersymmetric sector), related to F-terms of the form $\hat{\mathcal{W}}^{2g}$ where $\hat{\mathcal{W}}$ is now the $\mathcal{N} = 1$ gauge superfield. Again the BPS-nature of these couplings classically implies a (first order) holomorphicity condition for the F_g which is broken at the quantum level. The difference to the type II side is, however, that the corresponding (integrable) holomorphic anomaly equation, which captures this breaking, does no longer close on the class of functions F_g , but introduces new (semi)-topological objects. The latter have been understood to be physically related to BPS couplings of the form $\Pi^n \hat{\mathcal{W}}^{2g}$ where Π 's are chiral projections of non-holomorphic functions of chiral superfields.

More recently, after studying a particular world-sheet involution of a certain class of $\mathcal{N} = 4$ topological amplitudes [8, 9] (see also [10] as well as the review [11]), we have presented a completely new class of $\mathcal{N} = 2$ topological amplitudes [12]. The latter compute a corresponding class of 1/2-BPS terms in the low energy effective action of the form $\mathcal{F}_g K_-^{2g}$, where K_- is a particular superdescendant of the vector superfield in $\mathcal{N} = 2$ harmonic superspace [13]. The novel property is that their coupling coefficients depend on both holomorphic vector multiplet as well as particular (Grassmann) analytic hypermultiplet moduli, despite the common wisdom of mutual decoupling. Classically, this particular analytic dependence — which is again a direct consequence of the BPS nature of the couplings — can be captured by differential equations, namely a *holomorphicity* relation for the vector moduli and a *harmonicity relation* together with another second order equation for the hypermultiplet moduli. As before, at the quantum level these differential equations get modified by anomalous string world-sheet boundary contributions.

In this work, we show that these new topological objects are in fact a particular subclass of an even further generalized set of $\mathcal{N} = 2$ topological amplitudes. We make a systematic analysis of the latter on the heterotic string side (compactified on $K3 \times T^2$) by first computing them as genus- g amplitudes involving an appropriate number of fermions (gauginos and hyperinos) and establish explicitly their connection with the (semi)-topological theory obtained by twisting only the supersymmetric (left-moving) sector. We then show that this generic amplitude corresponds to a new series of higher order couplings in the effective action, involving both vector multiplets and neutral hypermultiplets. They depend on the moduli (both vectors and hypermultiplets) in a quite generic manner, which particularly means that

there is no immediate direct generalization of the holomorphicity or harmonicity equation. However, the BPS nature of these couplings manifests itself in another very useful property: The particular (Grassmann)-analytic projection which is necessary for consistency of these couplings leads us to establish relations between different component couplings. On the string theory side, these relations turn out to be highly non-trivial (first order) differential equations which again get modified by anomalous world-sheet boundary contributions. We explicitly compute these anomaly terms for these more general amplitudes (up to possible curvature dependent contact terms) as well as for the subclass considered in [12] and show that the equations close in the sense that no new topological objects appear. This indicates that the amplitude we consider captures the most generic expression in this particular class of BPS-couplings. We study the integrability of these equations and show that for the latter, the equations we have obtained are integrable including also the curvature terms.

The paper is organized as follows. In Section 2, we compute a particular multi-fermion genus- g physical amplitude and we show that it acquires a (semi)-topological expression as a correlation function in the twisted $K3 \times T^2$ heterotic σ -model. This amplitude has four types of indices, labeling the two different helicities (i.e. Weyl spinor components) of the antichiral gauginos and hyperinos involved. Contracting these indices (which have particular symmetry properties) with four fermionic variables, one can introduce a generating functional, in terms of which the results take a compact form and are greatly simplified. In Section 3, we describe the interpretation of the above amplitudes as a particular class of 1/2-BPS terms in the string effective action. In Sections 4 and 5, we derive differential equations for respectively the generalized amplitudes and the reduced ones that were obtained in [12]. As we show in Section 6, these relations follow from the particular (Grassmann)-analytic projection which we used in formulating the couplings. These equations are a direct consequence of the 1/2-BPS structure of these couplings and relate the anti-holomorphic dependence on the vector moduli with the dependence on the hypermultiplets of ‘wrong’ harmonicity in different component couplings. In string theory we find that (as usual) these equations get modified by an anomaly due to world-sheet boundary contributions. As a result, one obtains recursion relations for the non-holomorphic/harmonic moduli dependence of the above amplitudes. In Sections 4 and 5 we also study the integrability conditions of the anomaly equations. Our conclusions are presented in Section 7. Although the notation and conventions of this paper are those of Ref. [12], to make it self-contained, we include several appendices. Appendix A gives a brief review of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras, Appendix B summarizes the main properties of $\mathcal{N} = 2$ harmonic superspace, Appendix C presents some essential features of the heterotic $K3 \times T^2$ compactification, Appendix D describes the gauge freedom associated to one of the two general series of couplings, and Appendix E contains the direct string derivation of the differential equations, as well as the computation of a particular world-sheet boundary contribution which we deemed too lengthy to be presented in the main body of the paper.

2 Generalized $\mathcal{N} = 2$ Topological Amplitudes

2.1 A Further Extension of a Class of Topological Amplitudes

In the paper [12] by studying higher derivative couplings of holomorphic $\mathcal{N} = 2$ vector multiplets with hypermultiplets of a particular analyticity in heterotic string theory compactified on $K3 \times T^2$, a new class of $\mathcal{N} = 2$ topological amplitudes was discovered. They can be expressed as the following (semi-)topological correlators

$$\mathcal{F}^g = \int_{\mathcal{M}_g} \langle (\mu \cdot G_{T^2}^-)^g (\mu \cdot G_{K3,+}^-)^{2g-4} (\mu \cdot J_{K3}^-) \psi_3(\alpha) \rangle \cdot (\det Q_1)(\det Q_2). \quad (2.1)$$

Here $\int_{\mathcal{M}_g}$ denotes the integral over the moduli space of Riemann surfaces of genus g . Deformations of the latter are parametrized by a total of $3g - 3$ Beltrami differentials μ . The operators sewed with these Beltrami differentials in (2.1) are part of a twisted $\mathcal{N} = 2$ superconformal algebra, which we have briefly reviewed in appendix A. The free fermion ψ_3 , which is inserted at an arbitrary position α on the Riemann surface, serves the purpose of soaking up the holomorphic zero mode of the T^2 torus. Finally, $Q_{1,2}$ are the zero modes of the corresponding right-moving currents $\bar{J}_{1,2}$ associated with the gauge group in the heterotic theory.

The correlators (2.1), however, are not yet the most generic objects one might envisage. Indeed, already in [12] a generalization was worked out by allowing also a dependence on non-holomorphic vector multiplets. Indeed, the following semi-topological expression was found in a g -loop heterotic string amplitude for $n \geq 0$ (for details of the notation see [12])

$$\mathcal{F}_{g,n} = \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G_{T^2}^-)^{g-n} (\mu \cdot G_{K3}^-)^{2g+2n-4} (\mu \cdot J_{K3}^-) \prod_{j=1}^n \left[\int_{w_j} (\bar{\psi}_3 \bar{J})(J_{K3}^{++} \bar{J})(v_j) \right] \psi_3(p) (\det Q)^2 \rangle$$

where $\int_{\mathcal{M}_{(g,n)}}$ denotes the integral over the moduli space of a genus g Riemann surface with n punctures v_j . Note that deformations of the latter are parametrized by $3g - 3 + n$ Beltrami differentials μ . Moreover, it was also noted that the hypermultiplets of the 'wrong' analyticity (*i.e.* different than the one induced by the F-term like measure in harmonic superspace) are associated with operators of the form $(\Xi^A, \bar{\Xi}^A)$. The latter are labeled by an $Sp(2n)$ index A and transform as doublets under the $SU(2)$ current algebra. Besides they are $\mathcal{N} = 4$ primary operators in the sense that each of them is annihilated by half of the supercharges

$$\oint G_{K3,i}^+ \Xi^A = 0, \quad \text{and} \quad \oint G_{K3,i}^- \bar{\Xi}^A = 0. \quad (2.2)$$

These fields have left and right conformal dimension $(1/2, 1)$ and are $SU(2)$ highest weight states

$$\oint J_{K3}^{++} \Xi^A = \oint J_{K3}^{--} \bar{\Xi}^A = 0, \quad \text{and} \quad \oint J_{K3}^{++} \bar{\Xi}^A = \Xi^A, \quad (2.3)$$

with $U(1)$ charges ± 1 respectively.

This suggests that the most generic amplitude, with a dependence on holomorphic and anti-holomorphic vector multiplets and hypermultiplets of all analyticities, should structurally be of the form¹

$$\begin{aligned} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} &:= \frac{1}{(3g+n-3)!} \cdot \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G^-)^{3g+n-3} (\mu \cdot J_{K3}^-) \prod_{a=1}^{m_1} \int \Xi^{A_a} \cdot \\ &\cdot \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle. \end{aligned} \quad (2.4)$$

with $g \geq n \geq m_1$ and $n > m_2$ arbitrary integers. As we will see in the remainder of this work, this is essentially correct and we will prove it by showing that there is indeed a string amplitude, which (after some manipulations) gives rise to this expression. More specifically, we will calculate in the next subsection a heterotic g -loop amplitude with insertions of $2g$ chiral gauginos, $2n - m_1 - m_2$ anti-chiral gauginos, two chiral and $m_1 + m_2 + 2$ anti-chiral hyperfermions and further discuss its relation to (2.4).

2.2 Heterotic BPS-Saturated String Amplitudes

2.2.1 BPS-Saturated Amplitudes

Following the discussion in [12], we will compute the heterotic amplitude at a generic point in the moduli space of the $K3 \times T^2$ compactification. For a brief review of the notation as well as for explicit expressions of the vertex operators see appendix C. Here, for convenience, we just give table 1 containing the fermion charges with respect to space-time and torus fermions (bosonized in terms of scalars ϕ_1 , ϕ_2 and ϕ_3 respectively) and the H -charge of the vertex operators for the physical fields as well as the picture changing operators (PCO). The last column **RM** denotes the right moving (bosonic) piece of the vertex operator. We have not displayed the superghost part in the table, however, all matter fields are understood to be inserted in the $(-1/2)$ ghost picture, i.e. their vertices come with $e^{-\phi/2}$, while the PCO come with a factor of e^ϕ . The PCO's are arranged in such a way that $g + m_1 + m_2 - n$ of them at $r_a \in \{s_3\}$ contribute the torus part while $2g + 2n - m_1 - m_2$ of them at $r_a \in \{s_H\}$ the $K3$ part. In fact, by charge conservation, this is the only possible grouping of all PCOs — of course we still have to take into account all possible distributions of the total number $3g - 2 + n$ of the picture changing operators into these two classes. Notice moreover that we have put c -ghosts together with the vertex operators at the points v_j and t_k which means that the latter will not be integrated over the world-sheet. Instead, however, the genus g world-sheet will have $n + 1$ punctures.

With this table we can write down the amplitude in a straight-forward manner (for our notation concerning the spin structure dependence see appendix C.2). In order to save

¹Notice that we have turned the arbitrary position p here into an additional puncture v_{n+1} . Moreover, for reasons which will become clearer later on, we have denoted this expression with a tilde.

field	pos.	number	ϕ_1	ϕ_2	ϕ_3	H	RM
gaugino λ_-	x_i	g	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{\sqrt{2}}$	\bar{J}_{I_i}
	y_i	g	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{\sqrt{2}}$	\bar{J}_{J_i}
gaugino $\bar{\lambda}^+$	w_j	$n - m_1$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{\sqrt{2}}$	$\bar{J}_{\bar{I}_j}$
	v_j	$n - m_2$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{\sqrt{2}}$	$c\bar{J}_{\bar{J}_j}$
hyperino χ^{C_1}	z_1	1	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	$\bar{\Xi}^{C_1}$
hyperino χ^{C_2}	z_2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\bar{\Xi}^{C_2}$
hyperino $\bar{\psi}^{A_k}$	u_k	$m_1 + 1$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	$\bar{\Xi}^{A_k}$
hyperino $\bar{\psi}^{B_k}$	t_k	$m_2 + 1$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	$c\bar{\Xi}^{B_k}$
PCO	$\{s_3\}$	$g + m_1 + m_2 - n$	0	0	-1	0	∂X_3
	$\{s_H\}$	$2g + 2n - m_1 - m_2$	0	0	0	$-\frac{1}{\sqrt{2}}$	G_{K3}^-

Table 1: Fermion and H -charges for all vertex operators and picture changing operators (PCO). All physical vertex operators are inserted in the $-\frac{1}{2}$ picture, while all PCO contribute a factor of e^ϕ .

writing we define the following world-sheet positions

$$\begin{aligned}
a_1 &= \frac{1}{2} \sum_{i=1}^g (x_i - y_i) - \frac{1}{2} \sum_{j=1}^{n-m_1} w_j + \frac{1}{2} \sum_{j=1}^{n-m_2} v_j + \frac{1}{2} (z_1 - z_2) + \frac{1}{2} \sum_{k=1}^{m_2+1} t_k - \frac{1}{2} \sum_{k=1}^{m_1+1} u_k, \\
a_2 &= \frac{1}{2} \sum_{i=1}^g (x_i + y_i) - \frac{1}{2} \sum_{j=1}^{n-m_1} w_j - \frac{1}{2} \sum_{j=1}^{n-m_2} v_j - \frac{1}{2} (z_1 + z_2) + \frac{1}{2} \sum_{k=1}^{m_2+1} t_k + \frac{1}{2} \sum_{k=1}^{m_1+1} u_k - \sum_a^{\{s_3\}} r_a, \\
a_3 &= \frac{1}{\sqrt{2}} \sum_{i=1}^g (x_i + y_i) + \frac{1}{\sqrt{2}} \sum_{j=1}^{n-m_1} w_j + \frac{1}{\sqrt{2}} \sum_{j=1}^{n-m_2} v_j - \frac{1}{\sqrt{2}} \sum_a^{\{s_H\}} r_a,
\end{aligned}$$

with which the amplitude can be written in the following form

$$\begin{aligned}
\mathcal{A}_{I_1 \dots I_g, J_1 \dots J_g, \bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, C_1 C_2, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= F_{\{\Lambda\}, s}(a_1, a_2, a_3) G_{\{\Lambda\}}(x_i, y_i, w_j, v_j, z_1, z_2, t_k, u_k, \{s_3\}, \{s_H\}) \cdot \\
&\cdot \frac{\vartheta_s \left(\frac{1}{2} \left(\sum_i^g (x_i - y_i) + \sum_j^{n-m_1} w_j - \sum_j^{n-m_2} v_j + z_1 - z_2 - \sum_k^{m_2+1} t_k + \sum_k^{m_1+1} u_k \right) \right)}{\vartheta_s \left(\frac{1}{2} \left(\sum_i^g (x_i + y_i) + \sum_j^{n-m_1} w_j + \sum_j^{n-m_2} v_j + z_1 + z_2 + \sum_k^{m_2+1} t_k + \sum_k^{m_1+1} u_k \right) - \sum_a^{\{s_3, s_H\}} r_a - 2\Delta \right)} \\
&\cdot \frac{\prod_{i < j}^g E(x_i, x_j) E(y_i, y_j) \prod_{i < j}^{n-m_1} E(w_i, w_j) \prod_{i < j}^{n-m_2} E(v_i, v_j) \prod_{i < j}^{m_2+1} E(t_i, t_j) \prod_{i < j}^{m_1+1} E(u_i, u_j)}{\prod_a^{\{s_3, s_H\}} E(r_a, r_b) \prod_i^g E(x_i, z_2) E(y_i, z_1) \prod_j^{n-m_1} \prod_k^{m_2+1} E(w_j, t_k) \prod_j^{n-m_2} \prod_k^{m_1+1} E(v_j, u_k)}
\end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\prod_a^{\{s_3\}} E(r_a, z_1) E(r_a, z_2) \prod_j^{n-m_1} E(r_a, w_j) \prod_j^{n-m_2} E(r_a, v_j) \prod_a^{\{s_H\}} E^{\frac{1}{2}}(r_a, z_1) E^{\frac{1}{2}}(r_a, z_2)}{E^{\frac{1}{2}}(z_1, z_2) \prod_k^{m_2+1} E^{\frac{1}{2}}(z_1, t_k) E^{\frac{1}{2}}(z_2, t_k) \prod_k^{m_1+1} E^{\frac{1}{2}}(z_1, u_k) E^{\frac{1}{2}}(z_2, u_k) \prod_j^{m_2+1} E^{\frac{1}{2}}(t_j, u_k)} \\
& \cdot \frac{\prod_a^{\{s_H\}} \prod_k^{m_2+1} E^{\frac{1}{2}}(r_a, t_k) \prod_k^{m_1+1} E^{\frac{1}{2}}(r_a, u_k)}{\prod_{a < b}^{\{s_3\}} E(r_a, r_b) \prod_{a < b}^{\{s_H\}} E(r_a, r_b)} \cdot \langle \prod_i^g \bar{J}_{I_i} \bar{J}_{J_i} \prod_j^{n-m_1} \bar{J}_{\bar{I}_j} \prod_j^{n-m_2} \bar{J}_{\bar{J}_j} \rangle \\
& \cdot \langle \prod_a^{\{s_H\}} G_{K3,+}^-(r_a) \Xi^{C_1}(z_1) \Xi^{C_2}(z_2) \prod_k^{m_2+1} \Xi^{B_k}(t_k) \prod_k^{m_1+1} \Xi^{A_k}(u_k) \prod_{k=1}^{m_2+1} c(t_k) \prod_j^{n-m_2} c(v_j) \rangle. \quad (2.5)
\end{aligned}$$

For simplicity we will consider in the following differences of vector multiplet gauge groups in which case all of the operators \bar{J}_I and \bar{J}_J with holomorphic indices can only contribute zero modes. Since therefore their contribution is trivial, we will mostly drop them in the following. Furthermore, we are still free to choose a gauge and by picking the condition

$$\sum_a^{\{s_3\}} r_a + \sum_a^{\{s_H\}} r_a = \sum_{i=1}^g y_i + \sum_{j=1}^{n-m_2} v_j + z_2 + \sum_{k=1}^{m_2+1} t_k - 2\Delta, \quad (2.6)$$

essentially both the ϑ -functions cancel in (2.5). For the remaining expression we can directly perform the spin-structure sum using (C.6) with the result that $F_{\{\Lambda\},s}(a_1, a_2, a_3)$ is replaced by $F_{\{\Lambda\}}(a'_1, a'_2, a'_3)$ where, after using the gauge condition (2.6), we obtain

$$a'_1 = \sum_{i=1}^g x_i - z_2 - \Delta, \quad a'_2 = \sum_a^{\{s_3\}} r_a + \sum_{j=1}^{n-m_1} w_j - \sum_{k=1}^{m_2+1} t_k - \Delta, \quad (2.7)$$

$$a'_3 = \frac{1}{\sqrt{2}} \left(\sum_a^{\{s_4\}} r_a - 2 \sum_{j=1}^{n-m_2} v_j - z_1 - z_2 - \sum_{k=1}^{m_2+1} t_k + \sum_{k=1}^{m_1+1} u_k - 2\Delta \right). \quad (2.8)$$

Using the same manipulations as in [12] we can bring the amplitude into the following form

$$\begin{aligned}
\mathcal{A}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, C_1 C_2, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \int_{\mathcal{M}_{(g,n)}} (\mu \cdot b)^{3g-2+n} \frac{\det \omega_i(x_j) \det \omega_i(y_j)}{(\det(\text{Im} \tau))^2} \\
& \cdot \frac{\langle \prod_a^{\{s_H\}} G_{K3,+}^-(r_a) \Xi^{C_1}(z_1) \Xi^{C_2}(z_2) \prod_k^{m_2+1} c \Xi^{B_k}(t_k) \prod_k^{m_1+1} \Xi^{A_k}(u_k) \prod_j^{n-m_2} c e^{i\sqrt{2}H}(v_j) \rangle_{K3}}{\langle \prod_a^{3g+n} b(r_a) \prod_j^{n-m_2} c(v_j) c(z_1) c(z_2) \prod_k^{m_2+1} c(t_k) \rangle_{bc}} \\
& \cdot \langle \prod_a^{\{s_3\}} G_{T^2}^-(r_a) \prod_j^{n-m_1} \bar{\psi}_3(w_j) \prod_k^{m_2+1} \psi_3(t_k) \rangle_{T^2} \cdot \langle \prod_j^{n-m_1} \bar{J}_{\bar{I}_j} \prod_j^{n-m_2} \bar{J}_{\bar{J}_j} \rangle.
\end{aligned}$$

We can further simplify this expression by collapsing two of the PCO-insertion points with z_1 and z_2 respectively. Just as in [12], this converts these insertions to (0)-picture hyperscalar vertex operators which we can write as covariant derivatives in the following manner

$$\mathcal{A}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, C_1 C_2, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \mathcal{D}_+^{C_1} \mathcal{D}_+^{C_2} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \quad (2.9)$$

operator	dimension	charge	statistics
Ξ^A	1	-1	fermionic
$\psi_3 \Xi^B$	0	+2	bosonic
$\bar{\psi}_3 \bar{J}_{\bar{I}}$	1	-1	fermionic
$J_{K3}^{++} \bar{J}_{\bar{J}}$	0	+2	bosonic

Table 2: Dimensions and $U(1)$ charges of the operator insertions in (2.10). The last column denotes whether the operator is of fermionic or bosonic nature.

Performing now also the integration over x_i and y_i gives a factor of $(\det(\text{Im}\tau))^2$ which cancels the one already present. The left-over expression therefore becomes

$$\begin{aligned}
\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \\
= \int_{\mathcal{M}_{(g, n+1)}} (\mu \cdot b)^{3g-2+n} \langle \prod_a^{\{s_3\}} G_{T^2}^-(r_a) \prod_j^{n-m_1} \bar{\psi}_3(w_j) \prod_k^{m_2+1} \psi_3(t_k) \rangle_{T^2} \cdot \langle \prod_i^g \bar{J}_{I_i} \bar{J}_{J_i} \prod_j^{n-m_1} \bar{J}_{\bar{I}_j} \prod_j^{n-m_2} \bar{J}_{\bar{J}_j} \rangle \\
\cdot \frac{\langle \prod_a^{2g+2n-m_1-m_2-2} G_{K3,+}^-(r_a) \prod_k^{m_2+1} c \Xi^{B_k}(t_k) \prod_k^{m_1+1} \Xi^{A_k}(u_k) \prod_j^{n-m_2} c e^{i\sqrt{2}H}(v_j) \rangle_{K3}}{\langle \prod_a^{3g+n} b(r_a) \prod_j^{n-m_2} c(v_j) \prod_k^{m_2+1} c(t_k) \rangle_{bc}}.
\end{aligned}$$

In a final step we can again follow [12] and commute the positions of the PCO to the Beltrami differentials yielding

$$\begin{aligned}
\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \frac{1}{(g+m_1+m_2-n)!(2g+2n-m_1-m_2-2)!} \cdot \\
\cdot \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-m_2-2} \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \cdot \\
\cdot \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \tag{2.10}
\end{aligned}$$

Here we have introduced numerical normalization factors which will turn out helpful in reducing unnecessary writing in some of the later computations. Since this topological expression has four different types of insertions we have compiled the dimensions and $U(1)$ charges for all of them in table 2 for the reader's convenience. Notice that all insertions with dimension 1 are integrated over the world-sheet, while those with dimension 0 are inserted at fixed points associated to the $n+1$ punctures. We also note that this is not quite the object anticipated in (2.4), however, we shall see in section 2.3 that indeed it is very closely related.

2.2.2 Notation and Simplifications

Due to the great number of insertions, (2.10) is unfortunately a rather complicated expression. We therefore would like to introduce certain simplifications (and shorthand notations) in order to save writing and moreover also make the computations more transparent. Let us first introduce the operator

$$G^- = G_{T^2}^- + G_{K3,+}^- . \quad (2.11)$$

Although this operator does not have a well defined harmonic $U(1)$ -charge, there is no actual inconsistency because of the following reason. Upon writing (2.10) with the help of (2.11)

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \\ &= \frac{1}{(3g+n-2)!} \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G^-)^{3g+n-2} \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle , \end{aligned} \quad (2.12)$$

it is clear that in order to be able to provide the correct number of contractions with $\bar{\psi}_3$ (and to soak up the torus zero mode) from the $3g+n-2$ operators G^- precisely $(g+m_1+m_2-n)$ have to contribute the $G_{T^2}^-$ piece. All the remaining ones have to contribute the $G_{K3,+}^-$ part which makes sure that $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ has exactly the correct harmonic charge $2g+2n-m_1-m_2-2$. Therefore, (2.11) provides us with a helpful shorthand notation without loosing any information about the amplitude (2.10).

Eq.(2.12) is in fact identical to the one appearing in [7] in the context of $\mathcal{N} = 1$ heterotic topological amplitudes where there are $n+1$ charge (-1) operators and $n+1$ charge $(+2)$ operators together with the required $(3g+n-2)$ G^- insertions folded with the Beltrami differentials. These charge (-1) and $(+2)$ operators correspond to $\mathcal{N} = 1$ anti-chiral matter multiplets (with two different helicities). When the $\mathcal{N} = 1$ theory is embedded inside an $\mathcal{N} = 2$ theory, the $\mathcal{N} = 1$ anti chiral multiplets can come from $\mathcal{N} = 2$ vector multiplets or the hypermultiplets. In (2.12), the $(n+1)$ charge (-1) (or charge $(+2)$) operators are correspondingly split into m_1+1 (or m_2+1) that come from hypermultiplets and $n-m_1$ (or $n-m_2$) that come from vector multiplets. In other words, in (2.12), the specific features of the $\mathcal{N} = 2$ theory have not been used apart from splitting the operators in terms of their vector or hyper origins. In the next subsection we will define certain reduced topological amplitudes that are obtained from (2.12) by using the special properties of the $\mathcal{N} = 2$ theory namely the $\mathcal{N} = 4$ world-sheet superconformal structure and as a result they will not have an $\mathcal{N} = 1$ counterpart.

In [7], it was also pointed out that this class of amplitudes extends to genus zero surfaces, where they correspond to Π^{n+1} terms. On genus zero, there must be a net charge $+3$ and three of the dimension zero operators are unintegrated which implies that the number of G^- is $n-2$. It is clear that also in the $\mathcal{N} = 2$ case under consideration we will have the genus zero counterpart of (2.12) with the number of G^- being $n-2$.

To proceed further, it will turn out to be very useful to write (2.12) in a totally integrated form. To this end, for $g > 1$, we will localize the $n+1$ operators G^- , which are sewed with

operator	dimension	charge	statistics
Ξ^A	1	-1	fermionic
Φ^B	1	+1	fermionic
$\bar{\psi}_3 \bar{J}_{\bar{I}}$	1	-1	fermionic
$G_{K3,+}^+ \bar{J}_{\bar{J}}$	1	+1	fermionic

Table 3: Dimensions and $U(1)$ charges of the operator insertions in (2.15). The last column denotes whether the operator is of fermionic or bosonic nature.

the Beltrami differentials corresponding to the punctures of the Riemann surface, with the unintegrated insertions.² Using the OPE relations

$$\oint G^- J_{K3}^{++} \bar{J}_{\bar{J}} = G_{K3,+}^+ \bar{J}_{\bar{J}}, \quad (2.13)$$

$$\oint G^-(\psi_3 \Xi^B) =: \Phi^B = \partial X_3 \Xi^B - \psi_3 \oint G_{K3,+}^- \Xi^B, \quad (2.14)$$

we can rewrite the topological amplitude (2.12) as

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \\ &= \frac{1}{(3g-3)!} \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-3} \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \prod_{b=1}^{m_2+1} \int \Phi^{B_b} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} \int G_{K3,+}^+ \bar{J}_{\bar{J}_d} \rangle. \end{aligned} \quad (2.15)$$

For convenience we have again listed dimensions and charges of all insertions of this expression in table 3. Notice that all operators have now dimension 1 and are integrated over the world-sheet as it is appropriate. Let us also remind again that the operator Φ^B does not have well defined harmonic $U(1)$ charge. However, just as before, after performing all contractions of the ψ_3 free fermions, only those contributions will survive in (2.15) which have a well defined charge.

2.2.3 Symmetry Properties of the Amplitude

For later use we will consider now the symmetry properties of the topological expression (2.10). First of all, we notice that the latter is anti-symmetric in each set of indices A , B , \bar{I} and \bar{J} separately. This is manifest in the expression (2.10) for the indices A and \bar{I} and follows from the fermionic properties of the operators Ξ^A and $\bar{\psi}_3$ respectively. As we can

²As mentioned before, for the case of $g = 1$ and $g = 0$ there must be respectively one and three unintegrated punctures. This means that in these cases we can convert only n and $n - 2$ charge +2 vertices into integrated form by localizing the G^- sewed with the corresponding Beltrami differentials.

see from the fully integrated expression (2.15), in fact the same is also true for the indices B and \bar{J} as it is expected from the physical component amplitude which we started from in section 2.2.1.

Although not manifestly visible from the expression (2.12), the amplitudes \mathcal{H}^g also have symmetries upon exchanging, say, \bar{I} with \bar{J} indices. To see this, we consider the expression (2.15) with integrated insertions and rewrite for example

$$G_{K3,+}^+ \bar{J}_{\bar{I}} = \oint (\partial X_3 J_{K3}^{++} - \psi_3 G_{K3,+}^+) (\bar{\psi}_3 \bar{J}_{\bar{I}}). \quad (2.16)$$

Deforming the contour integral the only non-vanishing residues are

$$\oint (\partial X_3 J_{K3}^{++} - \psi_3 G_{K3,+}^+) \bar{\Xi}^A = \partial X_3 \Xi^A - \psi_3 \oint G_{K3,+}^+ \bar{\Xi}^A = \Phi^A, \quad (2.17)$$

$$\oint (\partial X_3 J_{K3}^{++} - \psi_3 G_{K3,+}^+) (\bar{\psi}_3 \bar{J}_{\bar{I}}) = G_{K3,+}^+ \bar{J}_{\bar{I}}, \quad (2.18)$$

such that we immediately get

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \frac{1}{(3g-3)!} \int_{\mathcal{M}_g} \left\langle (\mu \cdot G^-)^{3g-3} \prod_{a=1}^{m_1} \int \bar{\Xi}^{[A_a} \int \Phi^{A_{m_1+1}] \prod_{b=1}^{m_2+1} \int \Phi^{B_b} \right. \\ &\cdot \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \int \bar{\psi}_3 \bar{J}_{\bar{J}_1} \prod_{d=2}^{n-m_2} \int G_{K3,+}^+ \bar{J}_{\bar{J}_d} \left. \right\rangle + \frac{1}{(3g-3)!} \int_{\mathcal{M}_g} \left\langle (\mu \cdot G^-)^{3g-3} \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \right. \\ &\cdot \prod_{b=1}^{m_2+1} \int \Phi^{B_b} \prod_{c=1}^{n-m_1-1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \int \bar{\psi}_3 \bar{J}_{\bar{J}_1} \int G_{K3,+}^+ \bar{J}_{\bar{I}_{n-m_1}} \prod_{d=2}^{n-m_2} \int G_{K3,+}^+ \bar{J}_{\bar{J}_d} \left. \right\rangle. \end{aligned} \quad (2.19)$$

We can rewrite the right hand side of this equation to obtain

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1, \bar{J}_2 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, A_{m_1+1} B_1 \dots B_{m_2+1}} + \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{I}_{n-m_1}, \bar{J}_1, \bar{J}_2 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \quad (2.20)$$

Notice that if we formally combine the indices (\bar{I}, A) into a new (multi) index M and similarly (\bar{J}, B) into N , the relation is simply the statement

$$\mathcal{H}_{[M_1 \dots M_n, N_1] \dots N_n}^g = 0, \quad (2.21)$$

which is the direct generalization of identity (3.7) in [7].

2.2.4 Generating Functional

In view of the symmetry properties derived in the previous section, we can introduce a shorthand notation which will allow us to greatly simplify many of the following computations. Indeed, observing the particular fermionic nature of all insertions in table 3 it seems appropriate to introduce four types of Grassmann valued quantities $(\vartheta^{\bar{I}}, \eta^{\bar{J}}, \chi_A, \xi_B)$ and to define

the following generating functional — just like in [7]

$$\mathbb{H}^g := \sum_{n=0}^{\infty} \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1+1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_{n-m_2}} \xi_{B_1} \dots \xi_{B_{m_2+1}}}{(n-m_1)!(n-m_2)!(m_1+1)!(m_2+1)!} \cdot \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \quad (2.22)$$

In this form, every summand in \mathbb{H}^g can be thought of as a differential form with respect to $(\vartheta^{\bar{I}}, \eta^{\bar{J}}, \chi_A, \xi_B)$ of degree $(n-m_1, n-m_2, m_1+1, m_2+1)$. In order to also capture the particular case $m_2 = 0$, which will be discussed in more detail in section 2.4.1, let us also introduce a reduced version of \mathbb{H}^g . By restricting to the term $m_2 = 0$ in the summation of (2.22) we can define

$$\mathbb{H}_{\text{red}}^g := \sum_{n=0}^{\infty} \sum_{m_1=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1+1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_n} \xi_B}{(n-m_1)! n! (m_1+1)!} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B}. \quad (2.23)$$

Notice that every summand of this expression can be interpreted as a differential form with respect to $(\vartheta^{\bar{I}}, \eta^{\bar{J}}, \chi_A, \xi_B)$ of degree $(n-m_1, n, m_1+1, 1)$.

In order to demonstrate the usefulness of this notation let us just note that the symmetry property (2.20) can be written more compactly as

$$\left[\vartheta^{\bar{I}} \frac{\partial}{\partial \eta^{\bar{I}}} + \chi_A \frac{\partial}{\partial \xi_A} \right] \mathbb{H}^g = 0, \quad \text{and} \quad \left[\vartheta^{\bar{I}} \frac{\partial}{\partial \eta^{\bar{I}}} + \chi_A \frac{\partial}{\partial \xi_A} \right] \mathbb{H}_{\text{red}}^g = 0. \quad (2.24)$$

Here we have used the fact that (2.20) is valid for every single summand in the sum over m_2 in (2.22). Therefore it follows that the equation holds for \mathbb{H}^g as well as for the reduced generating functional (2.23).

2.3 χ -Exactness and Potential $\tilde{\mathcal{H}}$

An observation which will turn out important for the remainder of this work is the fact that \mathbb{H}^g is in a certain sense an exact form with respect to χ_A . A first hint towards this is to realize that \mathbb{H}^g is closed, i.e.

$$\chi_A \mathcal{D}_+^A \mathbb{H}^g = 0. \quad (2.25)$$

To see this, we consider the anti-symmetrized derivative

$$\begin{aligned} \mathcal{D}_+^{[A_1} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+2}, B_1 \dots B_{m_2+1}} &= \frac{1}{(g+m_1+m_2-n)!(2g+2n-m_1-m_2-2)!} \cdot \\ &\cdot \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K^3, +}^-)^{2g+2n-m_1-m_2-2} \int \oint G_{K^3, +}^+ \Xi^{[A_1} \prod_{a=2}^{m_1+2} \int \Xi^{A_a]} \cdot \\ &\cdot \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.26)$$

Deforming the contour integral $\oint G_{K3,+}^+$, however, it produces no pole with any of the other operators and therefore vanishes. This proves relation (2.25), however, we can derive an even stronger statement by writing $G_{K3,+}^- = \oint G_{K3,+}^+ J_{K3}^-$ in (2.10) and deforming the contour integration

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= -\frac{1}{(g+m_1+m_2-n)!(2g+2n-m_1-m_2-2)!} \cdot \\ &\cdot \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-m_2-3} (\mu \cdot J_{K3}^-) \int \oint G_{K3,+}^+ \bar{\Xi}^{[A_1} \cdot \\ &\cdot \prod_{a=2}^{m_1+1} \int \bar{\Xi}^{A_a]} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.27)$$

The insertion of $\int \oint G_{K3,+}^+ \bar{\Xi}^{A_1}$ corresponds to a derivative with respect to $\mathcal{D}_+^{A_1}$ which we can pull out. Writing the remaining expression in terms of the G^- we find

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= -\frac{1}{(2g+2n-m_1-m_2-2)(3g+n-3)!} \mathcal{D}_+^{[A_1} \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G^-)^{3g+n-3} \cdot \\ &\cdot (\mu \cdot J_{K3}^-) \prod_{a=2}^{m_1+1} \int \bar{\Xi}^{A_a]} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.28)$$

Upon introducing the quantity (which is indeed exactly the object we had anticipated in (2.4))

$$\begin{aligned} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &:= \frac{1}{(3g+n-3)!} \cdot \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G^-)^{3g+n-3} (\mu \cdot J_{K3}^-) \prod_{a=1}^{m_1} \int \bar{\Xi}^{A_a} \cdot \\ &\cdot \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle. \end{aligned} \quad (2.29)$$

we obtain the relation

$$-(2g+2n-m_1-m_2-2) \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \mathcal{D}_+^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}], B_1 \dots B_{m_2+1}}. \quad (2.30)$$

Comparing, however, to (2.10) we notice that the prefactor on the left hand side is identical to the harmonic charge of \mathcal{H} (i.e. it counts the total number of $G_{K3,+}^+$ operators). We can thus replace it by

$$D_0 \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \mathcal{D}_+^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}], B_1 \dots B_{m_2+1}}, \quad (2.31)$$

where D_0 is given in (B.3). Introducing the generating functional

$$\tilde{\mathbb{H}}^g := \sum_{n=0}^{\infty} \sum_{\substack{m_1=0 \\ m_2=0}}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_{n-m_2}} \xi_{B_1} \dots \xi_{B_{m_2+1}}}{(n-m_1)!(n-m_2)!(m_1+1)!(m_2+1)!} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}},$$

equation (2.31) can be more compactly written in the following form

$$D_0 \mathbb{H}^g = \chi_A \mathcal{D}_+^A \tilde{\mathbb{H}}^g. \quad (2.32)$$

Notice that this relation is in a certain sense equivalent to stating that \mathbb{H}^g is an exact form as concerning the χ_A variables. One should realize that similar to \mathbb{H}^g in (2.22), also every summand of $\tilde{\mathbb{H}}^g$ can be interpreted as a differential form in $(\vartheta^{\bar{I}}, \eta^{\bar{J}}, \chi_A, \xi_B)$. The important difference, however, is that the corresponding degrees are $(n - m_1, n - m_2, m_1, m_2 + 1)$, i.e. the degree in χ_A is shifted by one relative to (2.22).

2.4 ξ -Exactness and the Particular Case $m_2 = 0$

The discussion of the previous sub-section focused entirely on the properties of \mathbb{H}^g as a function of the χ_A variables. Since in the amplitude computation of section 2.2.1 the indices A and B (and therefore also the variables χ_A and ξ_B) enter on the same footing, one would expect similar properties of \mathbb{H}^g also with respect to the ξ_B variables. This is essentially the case, although, as we will discover now, there is a slight subtlety. Let us again begin by proving that \mathbb{H}^g is closed with respect to ξ_B , i.e.

$$\xi_B \mathcal{D}_+^B \mathbb{H}^g = 0. \quad (2.33)$$

To this end, we consider the anti-symmetrized derivative

$$\begin{aligned} \mathcal{D}_+^{[B_1} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2+2}]} &= \frac{1}{(3g + n - 1)!} \cdot \int_{\mathcal{M}_{(g, n+2)}} \langle (\mu \cdot G^-)^{3g+n-1} \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \cdot \\ &\cdot \Xi^{[B_1} \prod_{b=2}^{m_2+2} (\psi_3 \Xi^{B_b])} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.34)$$

Here we have inserted the vertex operator in its un-integrated form at an additional puncture of the genus g Riemann surface and added a further Beltrami differential. Writing now $\Xi^{B_1} = \oint \bar{\psi}_3 (\psi_3 \Xi^{B_1})$ and deforming the $\bar{\psi}_3$ contour integral it produces no pole with any of the other insertions. Therefore this expression vanishes, which proves equation (2.33).

2.4.1 The Case $m_2 = 0$

In order to understand in how far \mathcal{H}^g is also an exact form with respect to ξ_A (in the sense of the previous section), we will first discuss the particular case $m_2 = 0$, which has some interesting properties on its own. The corresponding \mathcal{H} takes the form of

$$\begin{aligned} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B} &= \frac{1}{(g + m_1 - n)! (2g + 2n - m_1 - 2)!} \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1-n} \cdot \\ &\cdot (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-2} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \Xi^B \rangle. \end{aligned} \quad (2.35)$$

In writing this expression we have used the fact that the only left-over ψ_3 (originally situated at t_1) is not able to contract with any other operator but is forced to soak the torus zero mode. It can therefore be inserted at an arbitrary position on the world-sheet which we call α . We can now proceed to transform the Ξ^B insertion into its integrated form by localizing one of the Beltrami differentials

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B} = \frac{1}{(g+m_1-n)!(2g+2n-m_1-3)!} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1-n} \cdot (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-3} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \int \oint G_{K3,+}^- \Xi^B \rangle.$$

The insertion of $\oint G_{K3,+}^- \Xi^B$ is, however, just a zero picture vertex operator, which we can pull out in the form of a moduli derivative

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B} = \mathcal{D}_+^B \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}}, \quad (2.36)$$

with the new topological expression

$$\begin{aligned} \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}} &= \\ &= \frac{1}{(3g+n-3)!} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-3} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle = \\ &= \frac{1}{(3g-3)!} \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-3} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^n \int (G_{K3,+}^+ \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.37)$$

We can also find a similar interpretation for equations (2.36) and (2.45) involving the reduced generating functional. Indeed, equation (2.36) can be rewritten in the following form

$$\mathbb{H}_{\text{red}}^g = \xi_A \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \quad (2.38)$$

with the newly introduced potential

$$\hat{\mathbb{H}}_{\text{red}}^g := \sum_{n=0}^{\infty} \sum_{m_1=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1+1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_n}}{(n-m_1)! n! (m_1+1)!} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}}. \quad (2.39)$$

For completeness, let us also mention that $\mathbb{H}_{\text{red}}^g$ and $\hat{\mathbb{H}}_{\text{red}}^g$ are of course still exact with respect to χ_A in the following sense

$$D_0 \mathbb{H}_{\text{red}}^g = \chi_A \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g = \chi_A \xi_B \mathcal{D}_+^A \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g, \quad (2.40)$$

$$D_0 \hat{\mathbb{H}}_{\text{red}}^g = \chi_A \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g, \quad (2.41)$$

with the corresponding potentials

$$\tilde{\mathbb{H}}_{\text{red}}^g := \sum_{n=0}^{\infty} \sum_{m_1=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_n} \xi_B}{(n-m_1)! n! (m_1+1)!} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1}, B}, \quad (2.42)$$

$$\hat{\hat{\mathbb{H}}}_{\text{red}}^g := \sum_{n=0}^{\infty} \sum_{m_1=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_n}}{(n-m_1)! n! (m_1+1)!} \hat{\hat{\mathcal{H}}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1}}, \quad (2.43)$$

where we have introduced the topological object

$$\begin{aligned}\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1}} &= \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} (\mu \cdot J_{K3}^{--}) \psi_3(\alpha) \prod_{a=1}^{m_1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_c}) \rangle = \\ &= \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-4} (\mu \cdot J_{K3}^{--}) \psi_3(\alpha) \prod_{a=1}^{m_1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^n \int (G_{K3,+}^+ \bar{J}_{\bar{J}_c}) \rangle, \end{aligned} \quad (2.44)$$

which satisfies the following identity

$$- (2g + 2n - m_1 - m_2 - 3) \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}} = \mathcal{D}^{[A_1} \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_2 \dots A_{m_1+1}}] \quad (2.45)$$

The degrees of every single summand in (2.40) are respectively

$$\deg(\tilde{\mathbb{H}}_{\text{red}}^g) = (n - m_1, n - m_2, m_1, 1), \quad (2.46)$$

$$\deg(\hat{\mathbb{H}}_{\text{red}}^g) = (n - m_1, n - m_2, m_1 + 1, 0), \quad (2.47)$$

$$\deg(\hat{\mathbb{H}}_{\text{red}}^g) = (n - m_1, n - m_2, m_1, 0). \quad (2.48)$$

We remark that (2.44) is precisely the topological expression conjectured in (2.4). Hence we have shown that there is also an actual string amplitude related to (2.44). This raises the question what the corresponding term in the string effective action looks like and whether it is possible to extract from it information about the 1/2-BPS nature of $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$ along the lines of [12, 8, 11]. We will return to this question in section 3.

2.4.2 The Case $m_2 > 0$ and Gauge Freedom

It is important to point out that all manipulations of the previous section were only possible for $m_2 = 0$ since only then the string theory amplitude allows to put the ψ_3 insertion at some arbitrary position. In the case of $m_2 > 0$, there would be additional contractions which invalidate these steps. However, independent of its origin in string theory, even for $m_2 > 0$ we can consider the following object

$$\begin{aligned}\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\alpha) &= \\ \frac{1}{(3g + n - 3)!} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-3} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^{m_2} (\psi_3 \Xi^{B_c}) \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.49)$$

Notice in particular that this quantity depends on the position α (which is a clear indication that it cannot come from a string theory computation). However, as we show in appendix D, changing the position α just amounts to adding an expression which is a derivative with respect to $\xi_B \mathcal{D}_+^B$

$$\hat{\mathbb{H}}^g(\alpha) - \hat{\mathbb{H}}^g(\beta) = \xi_B \mathcal{D}_+^B \mathbb{G}^g(\alpha, \beta), \quad (2.50)$$

where $\mathbb{G}^g(\alpha, \beta)$ is written in (D.4). Therefore, taking an anti-symmetrized derivative of $\hat{\mathcal{H}}$ we in fact obtain

$$\begin{aligned} \mathcal{D}_+^{[B_1] \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2+1}}}(\alpha) &= \frac{1}{(3g+n-2)!} \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G^-)^{3g+n-2} \psi_3(\alpha) \prod_{a=1}^{m_1+1} \int \Xi^{A_a} \cdot \\ &\cdot \prod_{b=1}^{n-m_1} \int (\bar{\psi}_3 \bar{J}_{\bar{I}_b}) \Xi^{[B_1]} \prod_{c=2}^{m_2+1} (\psi_3 \Xi^{B_c])} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (2.51)$$

Notice that this expression is independent of α : According to (2.50) changing the position of α results in an extra contribution which lies in the kernel of $\xi_B \mathcal{D}_+^B$. Thus we can put α at the position of Ξ^{B_1} , such that

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \mathcal{D}_+^{[B_1] \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2+1}}}(\alpha), \quad (2.52)$$

which in the language of the generating functionals is written as

$$\mathbb{H}^g = \xi_B \mathcal{D}_+^B \hat{\mathbb{H}}^g(\alpha). \quad (2.53)$$

Therefore, \mathbb{H}^g is also an exact form with respect to ξ , however, the corresponding potential has no nice interpretation within the framework of BPS-saturated string amplitudes. We note in passing that the subtle difference between χ_A and ξ_A is due to the particular choice of setup (and gauge fixing condition) in section 2.2.1, where we have computed the topological amplitude.

3 Harmonic description and Effective Action Coupling

In the previous Section we have considered particular amplitudes in heterotic string theory which are captured by correlation functions in a twisted two-dimensional theory. In this Section we would like to understand which terms in the heterotic effective action these amplitudes correspond to and whether they have any interesting properties with respect to their moduli dependence. It turns out that the effective action is best formulated in $\mathcal{N} = 2$ harmonic superspace [13], for which we have reviewed our conventions and discussed the relevant Grassmann analytic on-shell superfields in appendix B.

Using the G-analytic superfields (B.7) and (B.16), we now want to construct effective action couplings which correspond to the superstring amplitudes computed in section 2.2.1. As it turns out, just as the amplitudes themselves were generalizations of the expressions in [12], also the corresponding effective action couplings will be a generalization of those discussed in [12]. The coupling discussed in the latter work was of the form

$$S = \int [d\zeta] (K_- \cdot K_-)^g F^{-2(g-2)}(q_{\hat{A}a}^+, u), \quad (3.1)$$

where we have introduced the G-analytic superspace measure

$$\int [d\zeta] := \int d^4x \, du \, (\bar{D}_+ \cdot \bar{D}_+) (D^- \cdot D^-) = \int d^4x \, du \, d^2\theta^+ d^2\bar{\theta}_-, \quad (3.2)$$

$K_- \cdot K_- \equiv K_-^\alpha \epsilon_{\alpha\beta} K_-^\beta$, and $\hat{A} = 1, \dots, n$ is an $SO(n)$ vector index labeling the coordinates of the coset of physical scalars (for a discussion see [12]), while $a = 1, 2$ is an $SU(2)$ index. For a closer description of these superfields see appendix B.2. Notice finally, the superfield K_-^α being a fermion, one needs a gauge group of sufficiently high rank, so that $(K_-^\alpha \epsilon_{\alpha\beta} K_-^\beta)^g \neq 0$. We will assume that this requirement is met.

We can generalize (3.1) in various ways. The most general is to allow the coupling function to depend on holomorphic W_I and anti-holomorphic vector multiplets $\bar{W}_{\bar{I}}$ as well as hypermultiplets of both analyticities i.e. $(q^+, \tilde{q}_-) \leftrightarrow q_a^+$ and $(q^-, \tilde{q}_+) \leftrightarrow q_a^-$, $a = 1, 2$. To make this coupling manifestly G-analytic, however, we need to project it with four spinor derivatives

$$(D_- \cdot D_-)(\bar{D}^+ \cdot \bar{D}^+) F^{-2(g+1)}(q^+, q^-, W, \bar{W}; u), \quad (3.3)$$

where we have a great number of possibilities to distribute the derivatives, as we shall encounter in section 6. Note also that the $U(1)$ charge of the function F has changed, to compensate for the extra charges brought along by the derivatives. In addition to (3.3) we can also put a number of fermionic factors such that the most generic coupling takes the form

$$S = \int [d\zeta] (K_- \cdot K_-)^{g-1} (D_- \cdot D_-)(\bar{D}^+ \cdot \bar{D}^+) \left[(\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{J}})^{m_1} (\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{I}})^{m_2} \cdot (\bar{K}_{(+)}^{+\bar{I}} \bar{K}_{(-)}^{+\bar{J}})^{n-m_1-m_2-1} H_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{J}_1 \dots \bar{J}_{n-m_2-1}}^{(2-d)A_1 \dots A_{m_1}, B_1 \dots B_{m_2}}(q^+, q^-, W, \bar{W}; u) \right], \quad (3.4)$$

where $d = 2g + 2n - m_1 - m_2 - 2$ and $2 - d$ is the total harmonic charge of the function F . As usual, the operator $(D_- \cdot D_-)(\bar{D}^+ \cdot \bar{D}^+)$ is the projector on the G-analytic superspace, in order to match the properties of the measure. Moreover, the newly introduced $\bar{\Psi}$ denotes the antichiral hyperino. Since we now have both types of hypermultiplets, G-analytic and anti-analytic, we can obtain the same hyperino in two ways,

$$\bar{\Psi}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}^+ q^- \quad \text{or} \quad \bar{\Psi}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}^- q^+, \quad \dot{\alpha} = (\pm) \quad (3.5)$$

(both superfields start with the same spinor component). The labels (\pm) in (3.4) indicate the helicities of the corresponding spinors. In addition, in order to match the string theory computation of the previous section, we make a distinction between the labels A, B of $\bar{\Psi}$ and \bar{I}, \bar{J} of \bar{K} , in association with the helicity, to indicate their symmetry properties. The origin of this helicity splitting of the coupling (3.4) can be understood as follows. Let us start by denoting the two antichiral fermions collectively as

$$\Phi_{\dot{\alpha}}^M = (\bar{\Psi}_{A\dot{\alpha}}, \bar{K}_{\dot{\alpha}}^{+\bar{I}}). \quad (3.6)$$

Then we can write the expression in the square brackets in (3.4) in the form

$$(\Phi^{M_1} \cdot \Phi^{M_2}) \dots (\Phi^{M_{n-2}} \cdot \Phi^{M_{n-1}}) F_{M_1 M_2 \dots M_{n-2} M_{n-1}}, \quad (3.7)$$

where

$$(\Phi^M \cdot \Phi^N) = (\Phi^N \cdot \Phi^M) = \epsilon^{\dot{\alpha}\dot{\beta}} \Phi_{\dot{\alpha}}^M \Phi_{\dot{\beta}}^N = \Phi_{(+)}^M \Phi_{(-)}^N - \Phi_{(-)}^M \Phi_{(+)}^N \quad (3.8)$$

is the Lorentz invariant contraction of two spinors. Here (\pm) denotes the two projections of the spinor index $\dot{\alpha} = ((+), (-))$, or two helicities. Note that the expression in (3.7) must have a total harmonic charge $-2g + 2$ to match the charge of the first line in (3.4).

The coefficient function H in (3.7) has two obvious symmetries. One of them reflects the $M \leftrightarrow N$ symmetry of the contraction in (3.8) (recall that Φ are fermions), the other corresponds to swapping two such contractions $(\Phi^{M_{k-1}} \cdot \Phi^{M_k})$ and $(\Phi^{M_{l-1}} \cdot \Phi^{M_l})$. A third, less obvious symmetry is due to the two-component spinor cyclic identity

$$(\Phi^M \cdot \Phi^N) \Phi_{\dot{\alpha}}^K + (\Phi^N \cdot \Phi^K) \Phi_{\dot{\alpha}}^M + (\Phi^K \cdot \Phi^M) \Phi_{\dot{\alpha}}^N = 0. \quad (3.9)$$

These symmetries do not make the rank $(n-1)$ tensor $H_{M_1 M_2 \dots M_{n-2} M_{n-1}}$ fully irreducible. The reduction is achieved by breaking up the contractions (3.8) in (3.7) into pieces, first according to the two ingredients $\bar{\Psi}, \bar{K}$ in (3.6), and then according to helicity. At the first step we get three types of contractions,

$$(\bar{\Psi} \cdot \bar{\Psi})^a (\bar{\Psi} \cdot \bar{K}^+)^b (\bar{K}^+ \cdot \bar{K}^+)^c F^{(d)}, \quad a + b + c = n - 1, \quad (3.10)$$

where $2n-2$ is the total number of fermions and $d = -2g + 2 - b - 2c$ is the harmonic charge of the function H . Assuming that $a \leq c$, i.e. $d \leq 3 - 2g - n$, and using the identity (3.9), we can rewrite all contractions of the first type as contractions of the second type. So, from now on we set $a = 0$, $b + c = n - 1$. Further, we split each contraction into helicity states according to (3.8), and expand the resulting binomials. This gives rise to a number of terms of the type in the square brackets in (3.4), with $b = m_1 + m_2$ and $c = n - m_1 - m_2 - 1$. In the process, each term of the new type picks a particular irreducible projection of the tensor $H_{M_1 M_2 \dots M_{n-2} M_{n-1}}$. As an example, consider the following simple case:

$$(\bar{\Psi}_A \cdot \bar{K}^{+\bar{I}})(\bar{\Psi}_B \cdot \bar{K}^{+\bar{J}}) H_{\bar{I}\bar{J}}^{AB}. \quad (3.11)$$

The function H is symmetric under the exchange of pairs of indices \bar{I}_A and \bar{J}_B . This symmetry can be realized in two ways, $H_{(\bar{I}\bar{J})}^{(AB)}$ and $H_{[\bar{I}\bar{J}]}^{[AB]}$, each of which is associated with a particular helicity distribution:

$$\begin{aligned} & \left[(\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{I}})(\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{J}}) + (\bar{\Psi}_{B(+)} \bar{K}_{(-)}^{+\bar{J}})(\bar{\Psi}_{A(-)} \bar{K}_{(+)}^{+\bar{I}}) \right] H_{(\bar{I}\bar{J})}^{(AB)} \\ & \left[(\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{I}})(\bar{\Psi}_{B(+)} \bar{K}_{(-)}^{+\bar{J}}) + (\bar{\Psi}_{A(-)} \bar{K}_{(+)}^{+\bar{I}})(\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{J}}) \right] H_{[\bar{I}\bar{J}]}^{[AB]}. \end{aligned} \quad (3.12)$$

4 Differential Equations from String Theory for $m_2 > 0$

In this section we will derive differential equations for the amplitudes $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ in (2.10) and its 'potential' $\tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$ with respect to the vector multiplet and hypermultiplet moduli. All the equations in this section will be true only modulo possible contact terms that could give rise to moduli space curvature terms. In fact we have not been able to find the structure of the contact terms in these equations that would satisfy

the integrability conditions in the most generic setting. Violation of the integrability conditions therefore turns out to be proportional to curvature terms. Even so the fact that the equations are integrable in the flat limit is highly non-trivial and this will be proven in Section (4.3). However in Section 5 we will obtain differential equations for the reduced topological amplitudes $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}$ where we have been able to get the correct contact terms so that the resulting equations are exactly integrable even in the the case of general curved moduli spaces.

Based on the relations found in [12], we start out by applying the 'harmonicity operator' to $\tilde{\mathbb{H}}^g$. As we will see, this equation will lead us directly to further interesting equations. In particular, since $\tilde{\mathbb{H}}^g$ is related to \mathbb{H}^g , application of a projected derivative $\chi_A \mathcal{D}_+^A$ will yield a (weaker) condition for \mathbb{H}^g .³

4.1 Differential Equation for $\tilde{\mathbb{H}}^g$

We will start by applying the harmonicity operator to the 'χ-potential' $\tilde{\mathcal{H}}_{\tilde{I}_1 \dots \tilde{I}_{n-m_1}, \tilde{J}_1 \dots \tilde{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$. The covariant form of the harmonicity operator was given in [12] to be of the form

$$\nabla_-^A = -\mathcal{D}_-^A D_0 - \mathcal{D}_+^A D_-^+ . \quad (4.1)$$

However, since $\tilde{\mathcal{H}}_{\tilde{I}_1 \dots \tilde{I}_{n-m_1}, \tilde{J}_1 \dots \tilde{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$ only depends on \bar{u}_+^i , we can also use the more practical expression involving — strictly speaking illegal — partial harmonic derivatives

$$\nabla_-^A = \epsilon^{ij} \frac{\partial}{\partial \bar{u}_+^i} \frac{\partial}{\partial f_j^A} . \quad (4.2)$$

Applying this operator to $\tilde{\mathbb{H}}^g$, we obtain in the language of the generating functionals the following equation (the explicit calculation is rather lengthy and technical and we have therefore relegated it to appendix E.2)

$$\left[(-D_0 + 2) \vartheta^{\tilde{I}} \mathcal{D}_{\tilde{I}} - \chi_A \mathcal{D}_-^A D_0 - \chi_A \mathcal{D}_+^A D_-^+ \right] \tilde{\mathbb{H}}^g = (-D_0 + 2) \tilde{\mathcal{C}}^{\text{bdy}} , \quad (4.3)$$

where (as already mentioned in appendix E.2) we have converted the prefactor $(2g + 2n - m_1 - m_2 - 2)$, which accompanies the boundary terms, into the action of the operator $(-D_0 + 2)$. We should mention at this point, that this relation is correct up to possible contact terms which stem from two of the punctures coming close together. Since these terms are notoriously difficult to handle, they have not been considered. We note, however, for completeness, that these terms will correspond to curvature contributions and are a consequence of the fact that the moduli space of the heterotic compactification is not flat.

In fact, it will be much more convenient to slightly rewrite the harmonicity operator by moving the (projected) hyper-derivatives to directly hit $\tilde{\mathbb{H}}^g$

$$\left[(-D_0 + 2) \vartheta^{\tilde{I}} \mathcal{D}_{\tilde{I}} + (-D_0 + 2) \chi_A \mathcal{D}_-^A - D_-^+ \chi_A \mathcal{D}_+^A \right] \tilde{\mathbb{H}}^g = (-D_0 + 2) \tilde{\mathcal{C}}^{\text{bdy}} . \quad (4.4)$$

³Notice, since the kernel of $\chi_A \mathcal{D}_+^A$ is non-trivial, the equation for \mathbb{H}^g follows from the equation for $\tilde{\mathbb{H}}^g$ but not the other way round.

Using relation (2.32), we may further write

$$(-D_0 + 2) \left[\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \chi_A \mathcal{D}_-^A \right] \tilde{\mathbb{H}}^g + (-D_0 + 2) D_-^+ \mathbb{H}^g = (-D_0 + 2) \tilde{\mathbb{C}}^{\text{bdy}}. \quad (4.5)$$

Notice that here $(-D_0 + 2)$ acts on the whole equation. Since none of the topological amplitudes lies in the kernel of $(-D_0 + 2)$ we can drop it and formulate the *first order* differential equation

$$\mathfrak{D} \tilde{\mathbb{H}}^g + D_-^+ \mathbb{H}^g = \tilde{\mathbb{C}}^{\text{bdy}}, \quad (4.6)$$

where for later convenience we have introduced the first order differential operator

$$\mathfrak{D} := \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \chi_A \mathcal{D}_-^A, \quad (4.7)$$

and $\tilde{\mathbb{C}}^{\text{bdy}}$ as given in Appendix E is

$$\begin{aligned} \tilde{\mathbb{C}}^{\text{(bdy)}} = & \sum_{g_s=0}^g \left[\left(\mathcal{D}_+^A \tilde{\mathbb{H}}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) - \left(\mathcal{D}_+^A \mathbb{H}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \xi_B} \right) + \right. \\ & \left. + \left(\frac{\partial \mathbb{H}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + \left(\mathcal{D}_I \tilde{\mathbb{H}}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) - \left(\mathcal{D}_I \mathbb{H}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right]. \quad (4.8) \end{aligned}$$

4.2 Differential Equation for \mathbb{H}^g

After the discussion of $\tilde{\mathbb{H}}^g$ we will now also derive a differential equation for \mathbb{H}^g . We will show that combining the equations we have already obtained so far results in a new non-trivial differential equation for \mathbb{H}^g . In appendix E.3, we have shown that the same equation can also be obtained from a direct computation within string theory.

We will start from equation (4.6) and act with the differential $\chi_A \mathcal{D}_+^A$ on it. Inserting the explicit form of the boundary contribution (E.17) we get

$$\begin{aligned} \chi_C \mathcal{D}_+^C \left[\mathfrak{D} \tilde{\mathbb{H}}^g + D_-^+ \mathbb{H}^g \right] = & \\ = \chi_C \mathcal{D}_+^C \sum_{g_s=0}^g \left[\left(\mathcal{D}_+^A \tilde{\mathbb{H}}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) - \left(\mathcal{D}_+^A \mathbb{H}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \xi_B} \right) + \right. \\ & \left. + \left(\frac{\partial \mathbb{H}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + \left(\mathcal{D}_I \tilde{\mathbb{H}}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) - \left(\mathcal{D}_I \mathbb{H}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right]. \quad (4.9) \end{aligned}$$

Our strategy is now to commute the $\chi_C \mathcal{D}_+^C$ through until it hits either \mathbb{H}^g or $\tilde{\mathbb{H}}^g$ and then

use either relation (2.25) or (2.32). Using the definition (4.7) of the operator \mathfrak{D} , we find

$$\begin{aligned}
& -D_0 \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \mathbb{H}^g - \chi_A \mathcal{D}_-^A (D_0 - 1) \mathbb{H}^g + \chi_C \chi_A [\mathcal{D}_+^C, \mathcal{D}_-^A] \tilde{\mathbb{H}}^g - \chi_A \mathcal{D}_-^A \mathbb{H}^g = \\
& = \sum_{g_s=0}^g D_0 \left[(\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right] + \\
& + \sum_{g_s=0}^g \chi_C \left[([\mathcal{D}_+^C, \mathcal{D}_+^A] \tilde{\mathbb{H}}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) - ([\mathcal{D}_+^C, \mathcal{D}_+^A] \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \xi_B} \right) \right].
\end{aligned} \tag{4.10}$$

The commutator terms can be computed using (B.25) following the outline of quaternionic geometry in appendix B.3

$$\chi_C \chi_A [\mathcal{D}_+^C, \mathcal{D}_-^A] = -2\chi^2 R \Upsilon_0, \quad \text{and} \quad \chi_C [\mathcal{D}_+^C, \mathcal{D}_+^A] = 2\chi_C \Omega^{CA} \Upsilon_{++}, \tag{4.11}$$

where $\chi^2 := \chi_A \chi_B \Omega^{AB}$ and we have used that the contribution of the $Sp(n)$ curvature is symmetric in (AC) . Using moreover the identification $\Upsilon_0 = D_0$ and $\Upsilon_{++} = D_+^-$, we obtain the simpler relation⁴

$$\begin{aligned}
& -D_0 \left(\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \chi_A \mathcal{D}_-^A \right) \mathbb{H}^g - 2\chi^2 R D_0 \tilde{\mathbb{H}}^g = \\
& = \sum_{g_s=0}^g D_0 \left[(\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right].
\end{aligned} \tag{4.12}$$

Here we have also used that $D_+^- \mathbb{H}^g = D_+^- \tilde{\mathbb{H}}^g = 0$, since \mathbb{H}^g and $\tilde{\mathbb{H}}^g$ only depend on \bar{u}_+^i . As we can see, D_0 becomes an overall operator which can be dropped, thus implying the relation

$$\mathfrak{D} \mathbb{H}^g + 2\chi^2 R \tilde{\mathbb{H}}^g = - \sum_{g_s=0}^g \left[(\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right]. \tag{4.13}$$

We would like to point out that this is also a first order differential equation, just as (4.6). Even though we have neglected possible contact terms in (4.6), we have shown the additional curvature dependent term R in (4.13). This is because the derivation of the equation for $\hat{\mathbb{H}}$ starting from that for $\hat{\tilde{\mathbb{H}}}$ in Section 5, will follow the same steps as above and there we will indeed keep track of all the contact terms correctly.

4.3 Integrability

An important consistency check for our differential equations (in particular the boundary terms) is integrability. For simplicity we will consider here only equation (4.13) and check

⁴Notice that this identification is not in contradiction to our previous paper [12]: There the charge Υ_0 had been constrained by the presence of a G-analytic integral measure of the harmonic superspace to be slightly shifted with respect to D_0 . In the case at hand, due to the presence of the hypermultiplets with 'wrong' analyticity, the integral measure is no longer analytic and there is no constraint fixing Υ_0 . Thus we are free to identify the latter with D_0 .

its self-consistency by computing $\mathfrak{D}^2 \mathbb{H}^g$. Since the square of the operator \mathfrak{D} is given by

$$\mathfrak{D}^2 = \vartheta^{\bar{I}} \vartheta^{\bar{J}} [\mathcal{D}_{\bar{I}}, \mathcal{D}_{\bar{J}}] + \chi_A \chi_B [\mathcal{D}_-^A, \mathcal{D}_-^B] = 2\chi^2 R D_-^+, \quad (4.14)$$

and since in this Section we are ignoring curvature terms, the relation we have to prove is

$$0 \stackrel{?}{=} \mathfrak{D} \left[\sum_{g_s=0}^g \left((\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right) \right] \quad (4.15)$$

Ignoring the curvature terms we can further move \mathfrak{D} across various covariant derivatives which results in the following equation.

$$\begin{aligned} 0 \stackrel{?}{=} & \sum_{g_s=0}^g \left[(\mathcal{D}_+^A \mathfrak{D} \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) - (\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \left(\frac{\partial}{\partial \xi_B} \mathfrak{D} \mathbb{H}^{g-g_s} \right) \right] + \\ & + \sum_{g_s=0}^g \left[(\mathcal{D}_I \mathfrak{D} \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) - (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial}{\partial \eta^{\bar{J}}} \mathfrak{D} \mathbb{H}^{g-g_s} \right) \right]. \end{aligned} \quad (4.16)$$

In order to further streamline this expression, we will use equation (4.13)

$$\begin{aligned} 0 \stackrel{?}{=} & \sum_{g_s=0}^g \sum_{h_s=0}^{g_s} \mathcal{D}_+^A \left(\mathcal{D}_+^C \mathbb{H}^{h_s} \Omega_{CD} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \xi_D} + (\mathcal{D}_I \mathbb{H}^{h_s}) G^{I\bar{J}} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \eta^{\bar{J}}} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) \\ & - \sum_{g_s=0}^g \sum_{h_s=0}^{g-g_s} (\mathcal{D}_+^A \mathbb{H}^{g_s}) \Omega_{AB} \frac{\partial}{\partial \xi_B} \left[\left(\mathcal{D}_+^C \mathbb{H}^{h_s} \Omega_{CD} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \xi_D} + (\mathcal{D}_I \mathbb{H}^{h_s}) G^{I\bar{J}} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \eta^{\bar{J}}} \right) \right] \\ & + \sum_{g_s=0}^g \sum_{h_s=0}^{g_s} \left[\mathcal{D}_I \left(\mathcal{D}_+^A \mathbb{H}^{h_s} \Omega_{AB} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \xi_B} + (\mathcal{D}_J \mathbb{H}^{h_s}) G^{J\bar{K}} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \eta^{\bar{K}}} \right) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right] \\ & - \sum_{g_s=0}^g \sum_{h_s=0}^{g-g_s} \left[(\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \frac{\partial}{\partial \eta^{\bar{J}}} \left(\mathcal{D}_+^A \mathbb{H}^{h_s} \Omega_{AB} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \xi_B} + (\mathcal{D}_J \mathbb{H}^{h_s}) G^{J\bar{K}} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \eta^{\bar{K}}} \right) \right]. \end{aligned}$$

Explicitly expanding this expression (and using anti-symmetry of Ω_{AB}) we find the following terms

$$\begin{aligned} 0 \stackrel{?}{=} & \sum_{g_s=0}^g \sum_{h_s=0}^{g_s} \left[\mathcal{D}_+^C \mathbb{H}^{h_s} \Omega_{CD} \mathcal{D}_+^A \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \xi_D} \Omega_{AB} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} + \mathcal{D}_+^A \mathcal{D}_I \mathbb{H}^{h_s} G^{I\bar{J}} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \eta^{\bar{J}}} \Omega_{AB} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} + \right. \\ & + \mathcal{D}_I \mathbb{H}^{h_s} G^{I\bar{J}} \mathcal{D}_+^A \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \eta^{\bar{J}}} \Omega_{AB} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} + \mathcal{D}_I \mathcal{D}_+^A \mathbb{H}^{h_s} \Omega_{AB} \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \xi_B} G^{I\bar{J}} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} + \\ & + \mathcal{D}_+^A \mathbb{H}^{h_s} \Omega_{AB} \mathcal{D}_I \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \xi_B} G^{I\bar{J}} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} + \mathcal{D}_J \mathbb{H}^{h_s} G^{J\bar{K}} \mathcal{D}_I \frac{\partial \mathbb{H}^{g_s-h_s}}{\partial \eta^{\bar{K}}} G^{I\bar{J}} \frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \left. \right] - \\ & - \sum_{g_s=0}^g \sum_{h_s=0}^{g-g_s} \left[\mathcal{D}_+^A \mathbb{H}^{g_s} \Omega_{AB} \mathcal{D}_+^C \frac{\partial \mathbb{H}^{h_s}}{\partial \xi_B} \Omega_{CD} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \xi_D} + \mathcal{D}_+^A \mathbb{H}^{g_s} \Omega_{AB} \mathcal{D}_I \frac{\partial \mathbb{H}^{h_s}}{\partial \xi_B} G^{I\bar{J}} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \eta^{\bar{J}}} + \right. \end{aligned}$$

$$\begin{aligned}
& + \mathcal{D}_+^A \mathbb{H}^{g_s} \Omega_{AB} \mathcal{D}_I \mathbb{H}^{h_s} G^{I\bar{J}} \frac{\partial^2 \mathbb{H}^{g-g_s-h_s}}{\partial \xi_B \partial \eta^{\bar{J}}} + \mathcal{D}_I \mathbb{H}^{g_s} G^{I\bar{J}} \mathcal{D}_+^A \frac{\partial \mathbb{H}^{h_s}}{\partial \eta^{\bar{J}}} \Omega_{AB} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \xi_B} + \\
& + \mathcal{D}_I \mathbb{H}^{g_s} G^{I\bar{J}} \mathcal{D}_+^A \mathbb{H}^{h_s} \Omega_{AB} \frac{\partial^2 \mathbb{H}^{g-g_s-h_s}}{\partial \eta^{\bar{J}} \partial \xi_B} + \mathcal{D}_I \mathbb{H}^{g_s} G^{I\bar{J}} \mathcal{D}_J \frac{\partial \mathbb{H}^{h_s}}{\partial \eta^{\bar{J}}} G^{J\bar{K}} \frac{\partial \mathbb{H}^{g-g_s-h_s}}{\partial \eta^{\bar{K}}} \Big].
\end{aligned}$$

Indeed, upon reshuffling the sums over the genera, all the terms on the right hand side of this expression mutually cancel, which proves integrability of (4.13) (up to possible curvature terms).

5 Differential Equations from String Theory for $m_2 = 0$

Since the case $m_2 = 0$ is of particular interest, we will discuss the differential equations for $\hat{\mathbb{H}}_{\text{red}}^g$ and $\hat{\mathbb{H}}_{\text{red}}^g$ separately and show some interesting relations to the earlier work performed in [12]. Contrary to the previous Section, here we will keep all the curvature terms and show that the resulting equations are integrable.

5.1 Differential Equation for $\hat{\mathbb{H}}_{\text{red}}^g$

Generically, the case $m_2 = 0$ can be directly read off from the differential equations (E.24) and (E.16).⁵ For completeness, however, we will also work out the equations explicitly for the particular object $\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_m}$ defined in (2.44). Here, in order to save writing, we simply relabel m_1 by m . The explicit computation is very similar to the case $m_2 > 0$ and only some salient features are outlined in appendix E.4. The final answer can again be written in the language of the generating functionals

$$\left[(-D_0 + 2) \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} - \chi_A \mathcal{D}_-^A D_0 - \chi_A \mathcal{D}_+^A D_+ \right] \hat{\mathbb{H}}_{\text{red}}^g = (-D_0 + 2) \hat{\mathbb{C}}_{\text{bdy}}^g, \quad (5.1)$$

with the boundary contribution as given in (E.28). Following the same steps as in section 4.1, this equation can in fact be rewritten as

$$(-D_0 + 2) \left[\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \chi_A \mathcal{D}_-^A \right] \hat{\mathbb{H}}_{\text{red}}^g + (-D_0 + 2) D_-^+ \hat{\mathbb{H}}_{\text{red}}^g = (-D_0 + 2) \hat{\mathbb{C}}_{\text{bdy}}^g, \quad (5.2)$$

which entails the first order differential equation

$$\mathcal{D} \hat{\mathbb{H}}_{\text{red}}^g + D_-^+ \hat{\mathbb{H}}_{\text{red}}^g = \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right]. \quad (5.3)$$

⁵Care has to be taken, however, that certain contributions are only non-zero for the underlying surface being a torus or sphere. In this case, some insertions should be understood in the non-integrated form. For a discussion of this subtlety, see e.g. [12].

5.2 Differential Equation for $\hat{\mathbb{H}}_{\text{red}}^g$

As in the case $m_2 \neq 0$ the equations (2.41) and (5.3) put together, allow us to derive a further equation for $\hat{\mathbb{H}}_{\text{red}}^g$. To this end we act with $\chi_C \mathcal{D}_+^C$ on (5.3). Following the same procedures as in section 4 we obtain

$$\begin{aligned} D_0(\mathfrak{D}\hat{\mathbb{H}}_{\text{red}}^g) + 2\chi^2 R D_0 \hat{\mathbb{H}}_{\text{red}}^g &= -\chi_C \mathcal{D}_+^C \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] \\ &= \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B D_0 \hat{\mathbb{H}}_{\text{red}}^g - 2R\chi_B (D_+^{-} \hat{\mathbb{H}}_{\text{red}}^{g_s}) \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} - \right. \\ &\quad \left. - \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} - 2R\chi_A \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) (D_+^{-} \hat{\mathbb{H}}_{\text{red}}^{g-g_s}) \right]. \end{aligned}$$

Using the fact that $\hat{\mathbb{H}}_{\text{red}}^{g-g_s}$ is only a function of \bar{u}_+^i , we can simplify this expression

$$\begin{aligned} D_0(\mathfrak{D}\hat{\mathbb{H}}_{\text{red}}^g) + 2\chi^2 R D_0 \hat{\mathbb{H}}_{\text{red}}^g &= \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} (D_0 + 1) \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g - \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] \\ &= \frac{1}{2} D_0 \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g, \end{aligned}$$

which implies the differential equation

$$\mathfrak{D}\hat{\mathbb{H}}_{\text{red}}^g + 2\chi^2 R \hat{\mathbb{H}}_{\text{red}}^g = \frac{1}{2} \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g. \quad (5.4)$$

5.3 Comparison to Earlier Results

From the differential equations derived in the previous subsections we can make direct contact to similar equations worked out in [12]. The first equation derived in the latter work was a holomorphicity relation with respect to the vector-multiplet moduli. Translated into our language, this is the relation⁶

$$\vartheta^{\bar{I}} \partial_{\bar{I}} \hat{\mathbb{H}}_{\text{red}}^g = 0. \quad (5.5)$$

⁶In fact in [12] it was erroneously stated that there would be additional boundary contributions on the right hand side of this equation, which would come from the exchange of vector multiplets in the limit when the genus g Riemann surface factorizes into a genus $g-1$ surface and a torus. However, this contribution should be cancelled by a further exchange of a state of charge 3, which gives the same contribution but must appear with opposite sign. In fact we have checked that this (highly non-trivial) cancellation is indeed necessary to guarantee integrability (and thus consistency) of the differential equations derived in the previous section, as we shall see in the next subsection.

Indeed, this equation is precisely the term in (5.3) which is independent of χ_A

$$\left. \mathfrak{D} \hat{\mathbb{H}}_{\text{red}}^g + D_-^+ \hat{\mathbb{H}}_{\text{red}}^g \right|_{\chi_A=0} = \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] \Big|_{\chi_A=0}. \quad (5.6)$$

Here we have used that $\hat{\mathbb{H}}_{\text{red}}^g$ is at least linear in χ_A and the only part of \mathfrak{D} which is independent of χ_A is exactly the vector derivative, as in (5.5).

Moreover, a second order differential equation with respect to the hypermultiplet scalars has been derived in [12]. In preparation of reproducing this equation, we consider the commutator

$$\begin{aligned} [\mathfrak{D} \chi_A \mathcal{D}_+^A - \chi_A \mathcal{D}_+^A \mathfrak{D}] \hat{\mathbb{H}}_{\text{red}}^g &= \mathfrak{D} D_0 \hat{\mathbb{H}}_{\text{red}}^g + \chi_A \mathcal{D}_+^A (D_-^+ \hat{\mathbb{H}}_{\text{red}}^g) - \\ &\quad - \chi_C \mathcal{D}_+^C \sum_{g_s=1}^{g-1} \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] \\ &= (D_0 - 2) \mathfrak{D} \hat{\mathbb{H}}_{\text{red}}^g + 2\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \hat{\mathbb{H}}_{\text{red}}^g + \frac{1}{2} D_0 \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g = \\ &= 2\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \hat{\mathbb{H}}_{\text{red}}^g - 2\chi^2 R (D_0 - 2) \hat{\mathbb{H}}_{\text{red}}^g + \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g. \end{aligned} \quad (5.7)$$

Singling out the term which is independent of $\vartheta^{\bar{I}}$ we obtain

$$\left. [\mathfrak{D} \chi_A \mathcal{D}_+^A - \chi_A \mathcal{D}_+^A \mathfrak{D}] \hat{\mathbb{H}}_{\text{red}}^g \right|_{\vartheta^{\bar{I}}=0} = \left[-2\chi^2 R (D_0 - 2) \hat{\mathbb{H}}_{\text{red}}^g + \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g \right] \Big|_{\vartheta^{\bar{I}}=0}$$

which can equivalently be rewritten in the following form

$$\begin{aligned} \chi_A \chi_B [\mathcal{D}_+^A \mathcal{D}_-^B - \mathcal{D}_-^A \mathcal{D}_+^B] \hat{\mathbb{H}}_{\text{red}}^g \Big|_{\vartheta^{\bar{I}}=0} &= \left[2\chi^2 R (D_0 - 2) \hat{\mathbb{H}}_{\text{red}}^g - \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g \right] \Big|_{\vartheta^{\bar{I}}=0} \\ \epsilon^{ij} \chi_A \chi_B \mathcal{D}_i^A \mathcal{D}_j^B \hat{\mathbb{H}}_{\text{red}}^g \Big|_{\vartheta^{\bar{I}}=0} &= \left[2\chi^2 R (D_0 - 2) \hat{\mathbb{H}}_{\text{red}}^g - \sum_{g_s=1}^{g-1} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^g \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g \right] \Big|_{\vartheta^{\bar{I}}=0} \end{aligned} \quad (5.8)$$

This is precisely the equation derived in [12], including also the boundary terms. Notice particularly the operator $(D_0 - 2)$ in the first term on the right hand side. Translated to the setup in [12], this becomes $2(g - 1)$, which is exactly the prefactor found there.

5.4 Integrability

We would also like to briefly check integrability of the differential equations derived in the previous section. For simplicity we will focus on equation (5.4) for which we have to prove

$$\mathfrak{D}^2 \hat{\mathbb{H}}_{\text{red}}^g = 2\chi^2 R D_-^+ \hat{\mathbb{H}}_{\text{red}}^g \stackrel{?}{=} \mathfrak{D} \left[-2\chi^2 R \hat{\mathbb{H}}_{\text{red}}^g + \frac{1}{2} \sum_{g_s=0}^g \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right]. \quad (5.9)$$

For the first term on the right hand side we can use (5.3), while the remaining terms will also contain commutator terms of covariant derivatives. The latter will result in curvature contributions, in particular

$$[\mathfrak{D}, \mathcal{D}_+^A]^C{}_D = 2\chi_B [\Omega^{BA} \delta_D^C R D_0 - R^{(BAC)}{}_D + R (\Omega^{BC} \delta_D^A + \Omega^{AC} \delta_D^B)] \left(\chi_C \frac{\partial}{\partial \chi_D} + \xi_C \frac{\partial}{\partial \xi_D} \right). \quad (5.10)$$

After some algebra we get

$$\begin{aligned} 0 \stackrel{?}{=} & -2\chi^2 R \sum_{g_s=0}^g \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] + \\ & + \sum_{g_s=0}^g \mathcal{D}_+^A \left[-2\chi^2 R \hat{\mathbb{H}}_{\text{red}}^{g_s} + \frac{1}{2} \sum_{g'_s=0}^{g_s} \mathcal{D}_+^C \hat{\mathbb{H}}_{\text{red}}^{g'_s} \Omega_{CD} \mathcal{D}_+^D \hat{\mathbb{H}}_{\text{red}}^{g-g'_s} \right] \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} + \\ & + 2 \sum_{g_s=0}^g \left[\chi_E [\Omega^{EA} R D_0 \delta_D^C - R^{(EAC)}{}_D] + \Omega^{DA} R (\Omega^{EC} \delta_D^A + \Omega^{AC} \delta_D^B) \right] \cdot \\ & \cdot \left(\chi_C \frac{\partial}{\partial \chi_C} + \xi_C \frac{\partial}{\partial \xi_C} \right) \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s}. \end{aligned} \quad (5.11)$$

Using at this stage that \hat{H}_{red}^g is independent of ξ_A , we can streamline this condition in the following manner

$$\begin{aligned} 0 \stackrel{?}{=} & -2\chi^2 R \sum_{g_s=0}^g \left[\mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} + \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} + \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} - \right. \\ & \left. - \left(\frac{\partial \hat{\mathbb{H}}_{\text{red}}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s} \right] - \sum_{g_s=0}^g D_+^+ \hat{\mathbb{H}}_{\text{red}}^{g_s} \sum_{g'_s=0}^{g_s} \mathcal{D}_+^A \hat{\mathbb{H}}_{\text{red}}^{g-g'_s} \Omega_{AB} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^{g-g_s+g'_s}, \end{aligned}$$

which is indeed satisfied, thus entailing that equation (5.4) is integrable. Notice that integrability holds also including the curvature contributions, which lends strong evidence to the fact that we have indeed obtained the full boundary contribution including the contact terms. We should point out that for $m_2 = 0$ using the relation $\frac{\partial}{\partial \xi_A} \tilde{\mathbb{H}}|_{\xi=0} = \mathcal{D}_+^A \hat{\mathbb{H}}$ we can obtain the differential equation for former in terms of the equation for the latter (5.4). The

resulting equation has an additional curvature dependent term as compared to the one obtained from (4.6). Including this additional piece we can prove integrability for equations for \mathbb{H} and $\hat{\mathbb{H}}$ to the first order in ξ (i.e. for $m_2 = 0$) for a general curved moduli space. This indeed shows that some contact terms are missing in (4.6). While for $m_2 = 0$ this extra contact term can be deduced from (5.3), we have not been able to extend this to the general case $m_2 \neq 0$.

6 Field Theory Explanation of Differential Equations

After having derived several differential equations for the string theory amplitudes \mathbb{H}^g , $\tilde{\mathbb{H}}^g$, $\hat{\mathbb{H}}_{\text{red}}^g$ and $\hat{\tilde{\mathbb{H}}}_{\text{red}}^g$ we would also like to obtain a better understanding from a field theoretic point of view. In the past [8, 9, 11, 12] such equations could always be explained to root in particular analyticity properties of the effective action couplings with respect to moduli as well as harmonic coordinates. Put differently, these equations were a consequence of the fact that the couplings did not depend on all multiplets in a random manner. As explained in section 3, the situation now is slightly different in so far as, say, a generic $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ depends on all projections of the hypermultiplets and holomorphic and anti-holomorphic vector multiplets alike. Thus, all equations derived in the previous sections indeed will not reflect the analyticity properties of a single coupling, but rather encode relations between two or more couplings which are induced by supersymmetry. We will explain this idea in the following for equation (4.6) while a similar argumentation will also hold for the remaining equations.

6.1 Extracting the Component Coupling

As a starting point let us first translate equation (4.6) back from the language of generating functionals into the language of effective couplings. To this end we recall that the coefficient of $(\chi_A)^{m_1+1} (\xi_B)^{m_2+1} (\vartheta^{\bar{I}})^{n-m_1} (\eta^{\bar{J}})^{n-m_2}$ of the left hand side of (4.6) reads

$$\mathcal{D}_{[\bar{I}_1} \tilde{\mathcal{H}}_{\bar{I}_2 \dots \bar{I}_{n-m_1}], \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} + \mathcal{D}_{-}^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}], B_1 \dots B_{m_2+1}} + D_{-}^{+} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \quad (6.1)$$

We note in passing that this expression is consistent in so far as each of the terms has the same harmonic charge $-(2g + 2n - m_1 - m_2 - 4)$. We would now like to prove that this expression is zero.⁷ To this end we realize that (6.1) involves three different component couplings: All three of them stem from (3.4), however, differ in the way the $(D_{-} \cdot D_{-})(\bar{D}^{+} \cdot \bar{D}^{+})$ spinor derivatives are distributed and how the Grassmannian integral measure (see equation (3.2)) is used on the (master) coupling function $H_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{J}_1 \dots \bar{J}_{n-m_2-1}}^{(2-d), A_1 \dots A_{m_1}, B_1 \dots B_{m_2}}$ to single out a particular component term. To make things precise, we consider $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ to correspond

⁷In fact (4.6) predicts that it is equal to some boundary contribution which in field theory arises due to integrating out auxiliary fields as has been discussed in [9]. We will not attempt to do this explicitly here.

to the component coupling

$$\int d^4x \int du (K_- \cdot K_-)^g (D^- \cdot D^-) \left[(\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{J}})^{m_1} (\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{I}})^{m_2} (\bar{K}_{(+)}^{+\bar{I}} \bar{K}_{(-)}^{+\bar{J}})^{n-m_1-m_2-1} \cdot (\bar{D}^+ \bar{W}^{\bar{I}} \cdot \bar{D}^+ \bar{W}^{\bar{J}}) (\bar{D}_+ q_{A_1}^+ \cdot \bar{D}_+ q_{B_1}^+) \right] \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \Big|_{\theta=\bar{\theta}=0}, \quad (6.2)$$

where $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ is defined in terms of the ‘master’ coupling

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} := \frac{\partial^6 H_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_2 \dots \bar{J}_{n-m_2}}^{(2-d), A_2 \dots A_{m_1+1}, B_2 \dots B_{m_2+1}}}{\partial W^2 \partial \bar{W}^{\bar{I}_1} \partial \bar{W}^{\bar{J}_1} \partial q_{A_1}^+ \partial q_{B_1}^+}, \quad (6.3)$$

where an appropriate antisymmetrization of the indices is understood. Notice that the coupling $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ indeed has charge $-2g - 2n + m_1 + m_2 + 2$ as required. The two $\tilde{\mathcal{H}}$ -couplings in (6.1) now correspond to slightly different couplings with a slightly modified distribution of the spinor derivatives. To be concrete we may for example consider the following distributions

$$\int d^4x \int du \bar{D}_+^{\dot{\alpha}} (D^- \cdot D^-) \left[(K_- \cdot K_-)^g (\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{J}})^{m_1} (\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{I}})^{m_2} (\bar{K}_{(+)}^{+\bar{I}} \bar{K}_{(-)}^{+\bar{J}})^{n-m_1-m_2-1} \cdot (\bar{D}^+ q_{A_1}^- \cdot \bar{D}^+ \bar{W}^{\bar{J}}) (\bar{D}_+ q_{B_1}^+)_{\dot{\alpha}} \right] \tilde{\mathcal{H}}_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \Big|_{\theta=\bar{\theta}=0}, \quad (6.4)$$

$$\int d^4x \int du \bar{D}_+^{\dot{\alpha}} (D^- \cdot D^-) \left[(K_- \cdot K_-)^g (\bar{\Psi}_{A(+)} \bar{K}_{(-)}^{+\bar{J}})^{m_1} (\bar{\Psi}_{B(-)} \bar{K}_{(+)}^{+\bar{I}})^{m_2} (\bar{K}_{(+)}^{+\bar{I}} \bar{K}_{(-)}^{+\bar{J}})^{n-m_1-m_2-1} \cdot (\bar{D}^+ \bar{W}^{\bar{I}} \cdot \bar{D}^+ \bar{W}^{\bar{J}}) (\bar{D}_+ q_{B_1}^+)_{\dot{\alpha}} \right] \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \Big|_{\theta=\bar{\theta}=0}, \quad (6.5)$$

with the following relations to the ‘master’ coupling

$$\tilde{\mathcal{H}}_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} := \frac{\partial^5 H_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_2 \dots \bar{J}_{n-m_2}}^{(2-d), A_2 \dots A_{m_1+1}, B_2 \dots B_{m_2+1}}}{\partial W^2 \partial \bar{W}^{\bar{J}_1} \partial q_{A_1}^+ \partial q_{B_1}^+}, \quad (6.6)$$

$$\tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} := \frac{\partial^5 H_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_2 \dots \bar{J}_{n-m_2}}^{(2-d), A_2 \dots A_{m_1+1}, B_2 \dots B_{m_2+1}}}{\partial W^2 \partial \bar{W}^{\bar{I}_1} \partial \bar{W}^{\bar{J}_1} \partial q_{B_1}^+}. \quad (6.7)$$

These couplings have charges $-(2g + 2n - m_1 - m_2 - 4)$ and $-(2g + 2n - m_1 - m_2 - 3)$ respectively, which is indeed as required. We should note that all expressions are still arbitrary functions with respect to the scalar moduli f_i^A and φ , as well as the harmonic variables \bar{u}_{\pm}^i . Before proceeding, however, we notice an opportunity for simplifying our notation and saving a lot of writing: Since all superfields which appear in (6.2), (6.4) and (6.5) are fermionic in nature, all four sets of indices (A, B, \bar{I} and \bar{J}) will be separately antisymmetrized. We can thus, just as in section 2.2.4 introduce Grassmann valued expressions $(\chi_A, \xi_B, \vartheta^{\bar{I}}, \eta^{\bar{J}})$ and formulate everything in terms of generating functionals \mathbb{H}^g and $\tilde{\mathbb{H}}^g$.

6.2 Harmonic Gauge Fixing

Our next step after preparing the component couplings is to perform the harmonic u -integral which is most easily done after fixing the harmonic gauge freedom. In order to do this, we proceed in a similar fashion as in [12] and reduce all couplings to $SU(2)$ irreducible representations. The main novelty, however, is that, due to the dependence on the anti-G-analytic hypermultiplets q^- , \tilde{q}_+ , the couplings can have an arbitrary scalar dependence. Thus the generic gauge fixed form for the three couplings respectively will look like (as announced, we will be using the language of the generating functionals)

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \longrightarrow \left[\mathbb{H}^g = \mathfrak{H}_{(i_1 \dots i_d)}^g(\varphi, f_i) \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_d)} \right] \Big|_{\chi^{m_1+1} \vartheta^{n-m_1} \xi^{m_2+1} \eta^{n-m_2}}, \quad (6.8)$$

$$\tilde{\mathcal{H}}_{\bar{I}_2 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \longrightarrow \left[\tilde{\mathbb{H}}^g = \tilde{\mathfrak{H}}_{(i_1 \dots i_{d-2})}^g(\varphi, f_i) \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_{d-2})} \right] \Big|_{\chi^{m_1+1} \vartheta^{n-m_1-1} \xi^{m_2+1} \eta^{n-m_2}}, \quad (6.9)$$

$$\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} \longrightarrow \left[\tilde{\mathbb{H}}^g = \tilde{\mathfrak{H}}_{(i_1 \dots i_{d-1})}^g(\varphi, f_i) \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_{d-1})} \right] \Big|_{\chi^{m_1} \vartheta^{n-m_1} \xi^{m_2+1} \eta^{n-m_2}}, \quad (6.10)$$

where $\Big|_{\chi^{a_1} \vartheta^{a_2} \xi^{a_3} \eta^{a_4}}$ means extraction of the appropriate coefficient in the power series expansion in terms of the Grassmann variables. The precise power also determines the harmonic charge, *i.e.* $d = 2g + 2n - m_1 - m_2 - 2$. Notice that after taking into account the definition in terms of the master couplings (6.6) and (6.7), we may immediately state the relation

$$\left[\mathfrak{H}_{(i_1 \dots i_d)}^g = -\frac{1}{d} \chi_A \mathcal{D}_{(i_d}^A \tilde{\mathfrak{H}}_{i_1 \dots i_{d-1})}^g \right] \Big|_{\chi^{m_1+1} \vartheta^{n-m_1} \xi^{m_2+1} \eta^{n-m_2}}, \quad (6.11)$$

which is exactly equation (2.32) which we find in string theory. After these preparations we are now in a position to explain equation (4.6). To this end, we consider (the equation is understood to be the coefficient of $\chi^{m_1+1} \vartheta^{n-m_1} \xi^{m_2+1} \eta^{n-m_2}$ in the Grassmann expansion)

$$\begin{aligned} \chi_A \mathcal{D}_{-}^A \tilde{\mathbb{H}}^g &= \bar{u}_-^j \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_{d-1})} \chi_A \mathcal{D}_j^A \tilde{\mathfrak{H}}_{(i_1 \dots i_{d-1})}^g = \\ &= \chi_A \mathcal{D}_j^A \tilde{\mathfrak{H}}_{(i_1 \dots i_{d-1})}^g \left[\bar{u}_-^j \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_{d-1})} + \epsilon^{j(i_1} \bar{u}_+^{i_2)} \dots \bar{u}_+^{(i_{d-1})} \right] = \\ &= \frac{1}{d} \chi_A \mathcal{D}_{(i_1}^A \tilde{\mathfrak{H}}_{i_2 \dots i_d)}^g D_{-}^+ (\bar{u}_+^{i_1} \dots \bar{u}_+^{i_d}) + \chi_A \mathcal{D}^{A,j} \tilde{\mathfrak{H}}_{(j i_1 \dots i_{d-2})}^g \bar{u}_+^{i_1} \dots \bar{u}_+^{i_{d-2}}. \end{aligned} \quad (6.12)$$

Using at this stage relation (6.11) as well as (6.6) and (6.7) we may rewrite

$$\left[\chi_A \mathcal{D}_{-}^A \tilde{\mathbb{H}}^g = -D_{-}^+ \mathbb{H}^g - \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \tilde{\mathfrak{H}}_{(i_1 \dots i_{d-2})}^g \bar{u}_+^{(i_1)} \dots \bar{u}_+^{(i_{d-2})} \right] \Big|_{\chi^{m_1+1} \vartheta^{n-m_1} \xi^{m_2+1} \eta^{n-m_2}}. \quad (6.13)$$

In the last term, however, we recognize again (6.9), such that

$$\chi_A \mathcal{D}_{-}^A \tilde{\mathbb{H}}^g + D_{-}^+ \mathbb{H}^g + \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \tilde{\mathbb{H}}^g = 0, \quad (6.14)$$

which is exactly relation (6.1) that we wished to prove. Notice, if we treat the auxiliary fields properly (instead of putting them to zero in the on-shell approach we are using here), the right-hand side will be modified by additional boundary terms, which is indeed what we find from the explicit string computation in equation (4.6).

There is one further comment we would like to make: The crucial step in this derivation was the second line of (6.12). There we have used that a single \bar{u}^i_- combined with set of (totally symmetric) \bar{u}^i_+ gives rise to two irreducible $SU(2)$ tensor structures corresponding to the two terms in the square bracket of the second line in (6.12). Since both of them have different isospin, they cannot be related to each other and thus, from a harmonic- $SU(2)$ point of view, equation (6.1) decomposes into two distinct equations. This in fact can also be directly verified in (4.6) by hitting it with the two projection operators D_+^- and $-D_0 + 2 - D_-^+ D_+^-$ respectively. In the first case we obtain

$$D_+^- \left[\mathfrak{D} \tilde{\mathbb{H}}^g + D_-^+ \mathbb{H}^g \right] = D_+^- \tilde{\mathbb{C}}^{\text{bdy}} \quad (6.15)$$

$$\chi_A \mathcal{D}_+^A \tilde{\mathbb{H}}^g - D_0 \mathbb{H}^g = 0, \quad (6.16)$$

which is just equation (2.32). Here we have used that D_+^- annihilates \mathbb{H}^g , $\tilde{\mathbb{H}}^g$ as well as every term contained in $\tilde{\mathbb{C}}^{\text{bdy}}$. Applying the operator $-D_0 + 2 - D_-^+ D_+^-$, we in turn obtain

$$\begin{aligned} (-D_0 + 2 - D_-^+ D_+^-) \left[\mathfrak{D} \tilde{\mathbb{H}}^g + D_-^+ \mathbb{H}^g \right] &= (-D_0 + 2 - D_-^+ D_+^-) \tilde{\mathbb{C}}^{\text{bdy}} \\ (-D_0 + 2) \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} \tilde{\mathbb{H}}^g + \chi_A (-D_0 + 2) \mathcal{D}_-^A \tilde{\mathbb{H}}^g - \chi_A D_-^+ \mathcal{D}_+^A \tilde{\mathbb{H}}^g &= (-D_0 + 2) \tilde{\mathbb{C}}^{\text{bdy}} \\ \left[(-D_0 + 2) \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} - \chi_A \mathcal{D}_-^A D_0 - \chi_A \mathcal{D}_+^A D_-^+ \right] \tilde{\mathbb{H}}^g &= (-D_0 + 2) \tilde{\mathbb{C}}^{\text{bdy}}, \end{aligned} \quad (6.17)$$

which is just the original harmonicity equation (4.3). Thus the two $SU(2)$ structures of (6.12) indeed encapsulate all the relevant information about the couplings \mathbb{H}^g and $\tilde{\mathbb{H}}^g$.

7 Conclusions

In this work we have studied a generalization of the new class of topological amplitudes introduced in our previous work [12]. These amplitudes compute the coupling-coefficients of a very broad class of 1/2-BPS terms in the string effective action, which mix vector multiplets with neutral hypermultiplets in a non-trivial manner. In fact, we find a pair of (series of) amplitudes, which we combine into the generating functionals $(\mathbb{H}^g, \tilde{\mathbb{H}}^g)$. These two objects are related through a hyper-moduli derivative and $\tilde{\mathbb{H}}^g$ plays the role of 'gauge-potential' for \mathbb{H}^g . In particular the moduli dependence of these coupling functions satisfies a set of differential equations emerging from their 1/2-BPS structure, generalizing in a sense the holomorphic anomaly and harmonicity equations studied in [12, 9]. In fact, the analytic projection which we use to obtain a consistent BPS action coupling requires non-trivial relations between different terms at the component level. The latter predict highly non-trivial differential equations between different topological amplitudes which we have directly checked in string theory. The main novelty is that they are first order differential equations

(as compared to the second order relations derived in [12, 9]), however, mixing \mathbb{H}^g with $\hat{\mathbb{H}}^g$. As usual, we find that they get violated by world-sheet boundary contributions which we have evaluated explicitly. In addition, however, due to the fact that the hypermultiplet and vector multiplet moduli space of the $\mathcal{N} = 2$ string compactification is not flat, we also obtain contact terms, which turn out to be very difficult to control for the generic amplitudes. However, we were able to identify a particular subset of the series of topological amplitudes (denoted $(\hat{\mathbb{H}}_{\text{red}}^g, \hat{\mathbb{H}}_{\text{red}}^g)$) for which we managed to control the curvature contributions and prove integrability (and thus consistency) of our equations. We should also note that the equations close on the subset $(\hat{\mathbb{H}}_{\text{red}}^g, \hat{\mathbb{H}}_{\text{red}}^g)$ in the sense that no new classes of topological objects are introduced. Thus one might hope, taking into account the first order nature of these equations as well as its iterative structure, that it is possible to iteratively solve for $(\hat{\mathbb{H}}_{\text{red}}^g, \hat{\mathbb{H}}_{\text{red}}^g)$, using similar methods as in the case of pure vector multiplet dependence (see e.g. [3, 4, 5, 6]). In this way one might be able to obtain further insight into the structure of the moduli spaces of $\mathcal{N} = 2$ string compactifications. From a field theoretic point of view, the origin of the anomalous terms can probably be also understood via the process of integrating out auxiliary fields (for a discussion see [12]). It would be very interesting in the future to make this point more precise.

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A Superconformal Algebras

For heterotic string theory compactified on $K3 \times T^2$ the world-sheet theory is a product of an $\mathcal{N} = 2$ and an $\mathcal{N} = 4$ superconformal field theory representing T^2 and $K3$ respectively. In the following we will briefly review both theories mainly in order to fix our notation.

A.1 The $\mathcal{N} = 2$ Superconformal Algebra

The (untwisted) $\mathcal{N} = 2$ superconformal algebra (SCA) contains besides the energy momentum tensor T_{T^2} two supercurrents $G_{T^2}^{\pm}$ which are positively and negatively charged with respect to a $U(1)$ Kac-Moody current J_{T^2} . Here we have added the subscript T^2 to all operators in order to indicate that they represent the torus piece of the heterotic world-sheet

theory. The conformal weights of these operators are given by

$$h_{T_{T^2}} = 2, \quad h_{G_{T^2}^\pm} = \frac{3}{2}, \quad h_{J_{T^2}} = 1. \quad (\text{A.1})$$

The non-trivial operator product expansions (OPE) of these objects are given by

$$T_{T^2}(z)T_{T^2}(w) = \frac{2T_{T^2}(w)}{(z-w)^2} + \frac{\partial_w T_{T^2}(w)}{z-w}, \quad (\text{A.2})$$

$$T_{T^2}(z)G_{T^2}^\pm(w) = \frac{3G_{T^2}^\pm(w)}{2(z-w)^2} + \frac{\partial_w G_{T^2}^\pm(w)}{z-w}, \quad (\text{A.3})$$

$$T_{T^2}(z)J_{T^2}(w) = \frac{J_{T^2}(w)}{(z-w)^2} + \frac{\partial_w J_{T^2}(w)}{z-w}, \quad (\text{A.4})$$

$$G_{T^2}^+(z)G_{T^2}^-(w) = \frac{6}{(z-w)^3} + \frac{2J_{T^2}(w)}{(z-w)^2} + \frac{2T_{T^2}(w) + \partial_w J_{T^2}(w)}{z-w}, \quad (\text{A.5})$$

$$J_{T^2}(z)G_{T^2}^\pm(w) = \pm \frac{G_{T^2}^\pm(w)}{z-w}, \quad (\text{A.6})$$

$$J_{T^2}(z)J_{T^2}(w) = \frac{2}{(z-w)^2}. \quad (\text{A.7})$$

An explicit representation of the corresponding algebra in terms of a free complex boson X_3 and fermion ψ_3 living on T^2 can be written as

$$T_{T^2} = \frac{1}{2}\psi_3 \overset{\leftrightarrow}{\partial} \bar{\psi}_3 + \partial X_3 \partial \bar{X}_3, \quad G_{T^2}^- = \bar{\psi}_3 \partial X_3, \quad G_{T^2}^+ = \psi_3 \partial \bar{X}_3, \quad J_{T^2} = \psi_3 \bar{\psi}_3. \quad (\text{A.8})$$

A twisted version of the $\mathcal{N} = 2$ SCA is given by redefining the energy momentum tensor in the following manner

$$T_{T^2} \rightarrow T_{T^2} - \frac{1}{2}\partial J_{T^2}. \quad (\text{A.9})$$

This in particular has the effect of shifting the conformal weight of all operators by half their $U(1)$ charge. In this way, we find

$$h_{T_{T^2}}^{\text{twist}} = 2, \quad h_{G_{T^2}^-}^{\text{twist}} = 2, \quad h_{G_{T^2}^+}^{\text{twist}} = 1, \quad h_{J_{T^2}}^{\text{twist}} = 1. \quad (\text{A.10})$$

The new conformal weights of the supercurrents allow us to identify $G_{T^2}^+$ with the BRST operator of the twisted theory, while $G_{T^2}^-$ becomes the parametrization anti-ghost thereby defining the measure of the topological string.

A.2 The $\mathcal{N} = 4$ Superconformal Algebra

An $\mathcal{N} = 4$ SCFT contains the energy momentum tensor T_{K3} which is accompanied by two doublets of supercurrents $(G_{K3}^+, \tilde{G}_{K3}^-)$ and $(\tilde{G}_{K3}^+, G_{K3}^-)$, transforming under an $SU(2)$ Kac-Moody current algebra formed by $(J_{K3}^{\pm\pm}, J_{K3})$. Here, we have added the subscript $K3$ in

order to indicate that these operators represent the $K3$ -piece of the heterotic world-sheet theory. The conformal weights of these operators are

$$h_{T_{K3}} = 2, \quad h_{G_{K3}^\pm} = h_{\tilde{G}_{K3}^\pm} = \frac{3}{2}, \quad h_{J_{K3}^{\pm\pm}} = h_{J_{K3}} = 1. \quad (\text{A.11})$$

The non-trivial OPEs of these objects are given by

$$\begin{aligned} J_{K3}^{--}(z)G_{K3}^+(0) &\sim \frac{\tilde{G}_{K3}^-(0)}{z}, & J_{K3}^{++}(z)\tilde{G}_{K3}^-(0) &\sim -\frac{G_{K3}^+(0)}{z}, \\ J_{K3}^{++}(z)G_{K3}^-(0) &\sim \frac{\tilde{G}_{K3}^+(0)}{z}, & J_{K3}^{--}(z)\tilde{G}_{K3}^+(0) &\sim -\frac{G_{K3}^-(0)}{z}, \end{aligned}$$

$$\begin{aligned} G_{K3}^+(z)G_{K3}^-(0) &\sim \frac{J_{K3}(0)}{z^2} - \frac{T_{K3}^B(0) - \frac{1}{2}\partial J_{K3}(0)}{z}, \\ \tilde{G}_{K3}^+(z)\tilde{G}_{K3}^-(0) &\sim \frac{J_{K3}(0)}{z^2} - \frac{T_{K3}^B(0) - \frac{1}{2}\partial J_{K3}(0)}{z}, \\ \tilde{G}_{K3}^+(z)G_{K3}^+(0) &\sim \frac{2J_{K3}^{++}(0)}{z^2} + \frac{\partial J_{K3}^{++}(0)}{z}, \\ \tilde{G}_{K3}^-(z)G_{K3}^-(0) &\sim \frac{2J_{K3}^{--}(0)}{z^2} + \frac{\partial J_{K3}^{--}(0)}{z}, \end{aligned}$$

while for any operator O_{K3}^q of $U(1)$ charge q , one has:

$$J_{K3}(z)O_{K3}^q(0) \sim q \frac{O_{K3}^q(0)}{z}.$$

A representation in terms of free bosons $X_{4,5}$ and fermions $\psi_{4,5}$ living on a torus-orbifold realization of $K3$ is given by

$$T_{K3} = \partial X_4 \partial \bar{X}_4 + \partial X_5 \partial \bar{X}_5 + \frac{1}{2}(\psi_4 \overset{\leftrightarrow}{\partial} \bar{\psi}_4 + \psi_5 \overset{\leftrightarrow}{\partial} \bar{\psi}_5), \quad (\text{A.12})$$

$$J_{K3} = \psi_4 \bar{\psi}_4 + \psi_5 \bar{\psi}_5, \quad J_{K3}^{++} = \psi_4 \psi_5, \quad J_{K3}^{--} = \bar{\psi}_4 \bar{\psi}_5, \quad (\text{A.13})$$

$$G_{K3}^+ = \psi_4 \partial \bar{X}_4 + \psi_5 \partial \bar{X}_5, \quad \tilde{G}_{K3}^+ = -\psi_5 \partial X_4 + \psi_4 \partial X_5, \quad (\text{A.14})$$

$$G_{K3}^- = \bar{\psi}_4 \partial X_4 + \bar{\psi}_5 \partial X_5, \quad \tilde{G}_{K3}^- = -\bar{\psi}_5 \partial \bar{X}_4 + \bar{\psi}_4 \partial \bar{X}_5. \quad (\text{A.15})$$

The topologically twisted theory can be defined after specifying an $\mathcal{N} = 2$ subalgebra inside the $\mathcal{N} = 4$. We can then similarly redefine the energy momentum tensor

$$T_{K3} \rightarrow T_{K3} - \frac{1}{2}\partial J_{K3}. \quad (\text{A.16})$$

In this way, just as in the $\mathcal{N} = 2$ case, the conformal dimensions of all operators are shifted by half their charge with respect to J_{K3}

$$\begin{aligned} h_{T_{K3}}^{\text{twist}} &= 2, & h_{G_{K3}^\pm}^{\text{twist}} &= h_{\tilde{G}_{K3}^\pm}^{\text{twist}} = 2, & h_{G_{K3}^\pm}^{\text{twist}} &= h_{\tilde{G}_{K3}^\pm}^{\text{twist}} = 1, \\ h_{J_{K3}^\pm}^{\text{twist}} &= 2, & h_{J_{K3}^{++}}^{\text{twist}} &= 0, & h_{J_{K3}}^{\text{twist}} &= 1. \end{aligned} \quad (\text{A.17})$$

In view of the harmonic coordinates introduced in appendix B.1 we will rewrite the $\mathcal{N} = 4$ superconformal algebra in an $SU(2)$ covariant manner. To this end, we group the supercurrents into $SU(2)$ -doublets in the following manner

$$G_{K3,i}^+ \equiv \begin{pmatrix} \tilde{G}_{K3}^+ \\ G_{K3}^+ \end{pmatrix}, \quad \text{and} \quad G_{K3,i}^- \equiv \begin{pmatrix} G_{K3}^- \\ -\tilde{G}_{K3}^- \end{pmatrix}. \quad (\text{A.18})$$

For these doublets the OPE relations take the following more compact form

$$G_{K3,i}^+(z)G_{K3,j}^+(0) \sim \epsilon_{ij} \left(\frac{2J_{K3}^{++}(0)}{z^2} + \frac{\partial J_{K3}^{++}(0)}{z} \right), \quad (\text{A.19})$$

$$G_{K3,i}^-(z)G_{K3,j}^-(0) \sim \epsilon_{ij} \left(\frac{2J_{K3}^{--}(0)}{z^2} + \frac{\partial J_{K3}^{--}(0)}{z} \right), \quad (\text{A.20})$$

$$G_{K3,i}^+(z)G_{K3,j}^-(0) \sim -\epsilon_{ij} \left(\frac{J(0)}{z^2} - \frac{T_{K3}(0) - \frac{1}{2}\partial J(0)}{z} \right), \quad (\text{A.21})$$

$$J_{K3}^{++}(z)G_{K3,i}^+(0) \sim 0, \quad J_{K3}^{++}(z)G_{K3,i}^- \sim \frac{G_{K3,i}^+}{z}, \quad (\text{A.22})$$

$$J_{K3}^{--}(z)G_{K3,i}^+(0) \sim -\frac{G_i^-}{z}, \quad J_{K3}^{--}(z)G_{K3,i}^- \sim 0. \quad (\text{A.23})$$

We are now also free to project the $SU(2)$ -indices with harmonic variables to define the following operators

$$G_{K3,\pm}^+ = G_{K3,i}^+ \bar{u}_{\pm}^i, \quad \text{and} \quad G_{K3,\pm}^- = G_{K3,i}^- \bar{u}_{\pm}^i. \quad (\text{A.24})$$

B $\mathcal{N} = 2$ Harmonic Superspace

In this section of the appendix we discuss our conventions for the $\mathcal{N} = 2$ harmonic superspace and introduce the Grassmann analytic superfields which will be needed throughout this paper. Our conventions are identical to [12].

B.1 $SU(2)$ Harmonic Variables

Throughout this paper we consider $\mathcal{N} = 2$ supersymmetry in four dimensions whose automorphism group is $SU(2)$. We introduce harmonic variables [13] on the coset $SU(2)/U(1)$ in the form of matrices $(u_i^+, u_i^-) \in SU(2)$. They have an index $i = 1, 2$ transforming under the fundamental representation of $SU(2)$ and $U(1)$ charges ± 1 . Together with their complex conjugates $\bar{u}_+^i = \overline{(u_i^+)}$, $\bar{u}_-^i = \overline{(u_i^-)}$ they satisfy the unitarity relations

$$u_i^+ \bar{u}_+^i = u_i^- \bar{u}_-^i = 1, \quad u_i^+ \bar{u}_-^i = u_i^- \bar{u}_+^i = 0, \quad u_i^+ \bar{u}_+^j + u_i^- \bar{u}_-^j = \delta_i^j \quad (\text{B.1})$$

and the unit determinant condition

$$\epsilon^{ij} u_i^+ u_j^- = 1, \quad u_i^+ u_j^- - u_i^- u_j^+ = \epsilon_{ij}, \quad (\text{B.2})$$

(with $\epsilon^{12} = -\epsilon_{12} = 1$).

The harmonic functions have harmonic expansions homogeneous under the action of the subgroup $U(1)$. The harmonic expansions are organized in irreps of $SU(2)$, keeping the balance of projected indices so that the overall $U(1)$ charge is always the same. An example of a harmonic function is *e.g.* $f^+(u) = f^i u_i^+ + f^{ijk} u_i^+ u_j^+ u_k^- + \dots$. The first component in this expansion is a doublet of $SU(2)$. The higher components give rise to higher-dimensional irreps.

The harmonic derivatives can be viewed as the covariant derivatives on the harmonic coset $SU(2)/U(1)$, or equivalently, as the generators of the algebra of $SU(2)$ written in a $U(1)$ basis. This means that they are invariant under the left action of the group $SU(2)$, but covariant under the right action of the subgroup $U(1)$. They can be split into generators of the subalgebra $U(1)$:

$$D_0 = u_i^+ \frac{\partial}{\partial u_i^+} - \bar{u}_+^i \frac{\partial}{\partial \bar{u}_+^i} \quad (\text{B.3})$$

and of the coset:

$$D_+^- = u_i^- \frac{\partial}{\partial u_i^+} = \bar{u}_+^i \frac{\partial}{\partial \bar{u}_-^i}, \quad \text{and} \quad D_-^+ = u_i^+ \frac{\partial}{\partial u_i^-} = \bar{u}_-^i \frac{\partial}{\partial \bar{u}_+^i}. \quad (\text{B.4})$$

The harmonic derivatives are differential operators preserving the defining algebraic constraints (B.1) and (B.2).

The derivative (B.3) acts homogeneously on the harmonic functions. For instance, the function $f^+(u)$ above has $U(1)$ charge $+1$, hence

$$D_0 f^+(u) = f^+(u). \quad (\text{B.5})$$

The harmonic expansion of this function defines an infinitely reducible representation of $SU(2)$. It can be made irreducible by requiring that the raising operator D_-^+ annihilates the function:

$$D_-^+ f^+(u) = 0 \quad \Rightarrow \quad f^+(u) = f^i u_i^+. \quad (\text{B.6})$$

In other words, such a function is a highest-weight state of a doublet of $SU(2)$. The irreducibility condition (B.6) is also called a condition for harmonic (H-) analyticity.

B.2 Grassmann Analytic On-shell Superfields

The introduction of harmonic variables allows us to define ‘1/2-BPS short’ or Grassmann (G-) analytic superfields.⁸ They depend only on half of the Grassmann variables which can

⁸The notion of Grassmann analyticity (with breaking of the R symmetry) was first proposed in [14] in the context of the $\mathcal{N} = 2$ hypermultiplet. Later on it was made R-symmetry covariant in the framework of $\mathcal{N} = 2$ harmonic superspace in [13].

be chosen to be $\theta_\alpha^+ = \theta_\alpha^i u_i^+$ and $\bar{\theta}_-^{\dot{\alpha}} = \bar{u}_-^i \bar{\theta}_i^{\dot{\alpha}}$ with $\alpha = (\pm)$ ($\dot{\alpha} = (\pm)$) an (anti-)chiral spinor index. In this work we will encounter two different examples which we will briefly review now (for more information see e.g. [12]).

B.2.1 Linearized On-shell Hypermultiplet

The first superfield which we will discuss is the *linearized on-shell* hypermultiplet ($\mathcal{N} = 2$ matter multiplet)

$$q^+(x^\mu, \theta^+, \bar{\theta}_-, u) = f^i u_i^+ + \theta_\alpha^+ \chi^\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\theta}_-^{\dot{\alpha}} + \text{derivative terms.} \quad (\text{B.7})$$

Here f^i are the two complex scalars, χ^α and $\bar{\psi}_{\dot{\alpha}}$ are the two fermions of the on-shell multiplet. To exhibit manifest G-analyticity, one has to choose the appropriate analytic basis in superspace,

$$x^\mu \rightarrow x^\mu + i\theta^+ \sigma^\mu \bar{\theta}_+ - i\theta^- \sigma^\mu \bar{\theta}_-, \quad (\text{B.8})$$

analogous to the familiar chiral basis. Then q^+ satisfies the conditions

$$D_-^\alpha q^+(\theta^+, \bar{\theta}_-, u) = \bar{D}_\alpha^+ q^+(\theta^+, \bar{\theta}_-, u) = 0. \quad (\text{B.9})$$

Note that the harmonic dependence in (B.7) is cut down to linear. This is typical for on-shell multiplets which, in addition to the G-analyticity condition (B.9), also satisfy the H-analyticity condition

$$D_-^+ q^+(\theta^+, \bar{\theta}_-, u) = 0. \quad (\text{B.10})$$

Here the harmonic derivative is supersymmetrized by going to the manifestly G-analytic superspace coordinates (B.8). One can show that the ‘ultrashort’ on-shell superfield (B.7) is a solution to the simultaneous conditions for G- and H-analyticity [13, 15, 16].

Note that in the $\mathcal{N} = 2$ G-analytic superspace there exists a special conjugation \sim combining complex conjugation with a reflection on the harmonic coset, such that G-analyticity is preserved. In this sense we can define the conjugate hypermultiplet

$$\tilde{q}_-(x^\mu, \theta^+, \bar{\theta}_-, u) = \bar{f}_i \bar{u}_-^i + \bar{\theta}_-^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} + \psi^\alpha \theta_\alpha^+ + \text{derivative terms.} \quad (\text{B.11})$$

In what follows it will be convenient to combine the two versions of the hypermultiplet into a doublet of an external $SU(2)$ (not the R symmetry one), $(q^+, \tilde{q}_-) \leftrightarrow q_a^+, a = 1, 2$.

B.2.2 Linearized On-shell Vector Multiplet

Another example of a G-analytic superfield is the *linearized* on-shell vector multiplet. It is obtained from the off-shell chiral field strength

$$W(\theta_\alpha^i) = \varphi + \theta_\alpha^i \lambda_i^\alpha + \theta_\alpha^i \theta_\beta^j (\epsilon_{ij} F_{(+)}^{(\alpha\beta)} + \epsilon^{\alpha\beta} S_{(ij)}) . \quad (\text{B.12})$$

Here φ is the complex physical scalar and $F_{(+)}$ is the self-dual part of the gluon field strengths, while S is a triplet of auxiliary fields. On-shell the latter must vanish, hence the additional constraint

$$\epsilon_{\alpha\beta} D_i^\alpha D_j^\beta W = 0. \quad (\text{B.13})$$

Now, define the superfield (a superdescendant of W)

$$K_-^\alpha = D_-^\alpha W, \quad (\text{B.14})$$

where we have projected the $SU(2)$ index of D_i^α with the harmonic \bar{u}_-^i . This superfield is annihilated by half of the spinor derivatives and hence is 1/2 BPS short. Indeed, this is true for the projections $\bar{D}_{\dot{\beta}}^+$ since $\{\bar{D}^+, D_-\} = 0$ and $\bar{D}_{\dot{\beta}}^i W = 0$ (chirality). Further, hitting (B.14) with D_-^β we obtain zero as a consequence of the projection of the on-shell constraint (B.13) with $\bar{u}_-^i \bar{u}_-^j$. We conclude that K_-^α satisfies the same G-analyticity constraints as the on-shell hypermultiplet (B.9)

$$D_-^\beta K_-^\alpha = \bar{D}_{\dot{\beta}}^+ K_-^\alpha = 0, \quad (\text{B.15})$$

which imply that K_- depends only on half of the θ 's:

$$K_-^\alpha(\theta^+, \bar{\theta}_-, u) = \lambda_i^\alpha \bar{u}_-^i + (\sigma^\mu)^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}-} - i\partial_\mu \varphi + \theta_\beta^+ F_{(+)}^{\alpha\beta} + \text{derivative terms}. \quad (\text{B.16})$$

In addition, the harmonic dependence of K_-^α is restricted to be linear. As in (B.10), this follows from the condition for H-analyticity

$$D_-^+ K_-^\alpha = 0, \quad (\text{B.17})$$

in turn derived from the harmonic independence of W ($D_-^+ W = 0$) and the commutator $[D_-^+, D_-^\alpha] = 0$. This is another example of an ultrashort superfield. Note, however, that it is not a primary object but rather a superdescendant of the chiral on-shell vector multiplet.

B.3 Quaternionic Geometry

For completeness, we will review in this appendix some relevant aspects of quaternionic manifolds. We will closely follow the discussion in [17] and use the same notation and conventions as in [12]. Following [18, 19, 20], a quaternionic manifold \mathfrak{M} is defined as a $4n$ dimensional Riemann manifold whose holonomy group is restricted to a subgroup of $Sp(n) \times Sp(1)$. Therefore, we can locally choose a coordinate frame $\{x^{Mk}\}$ where $M = 1, \dots, 2n$ and $k = 1, 2$ are indices of $Sp(n)$ and $Sp(1) \sim SU(2)$ respectively. In this coordinate frame, the covariant derivatives can be written as

$$\begin{aligned} \mathcal{D}_{Ai} &= e_{Ai}^{Mk} \partial_{Mk} - \omega_{Ai(CD)} \Gamma^{(CD)} - \omega_{Ai(lk)} \Gamma^{(lk)} \\ &:= \nabla_{Ai} - \omega_{Ai(CD)} \Gamma^{(CD)} - \omega_{Ai(lk)} \Gamma^{(lk)}, \end{aligned} \quad (\text{B.18})$$

where we have introduced the $Sp(n)$ and $Sp(1)$ connections $\omega_{Ai(CD)}$ and $\omega_{Ai(lk)}$ respectively. Our convention for the $Sp(n)$ and $Sp(1)$ algebra of the generators $\Gamma^{(CD)}$ and $\Gamma^{(lk)}$ follows [17] and is given by

$$[\Gamma^{(CD)}, \Gamma^{(EF)}] = \frac{1}{2} (\Omega^{CE} \Gamma^{(DF)} + \Omega^{CF} \Gamma^{(DE)} + \Omega^{DE} \Gamma^{(CF)} + \Omega^{DF} \Gamma^{(CE)}), \quad (\text{B.19})$$

$$[\Gamma^{(lk)}, \Gamma^{(mn)}] = \frac{1}{2} (\epsilon^{lm} \Gamma^{(kn)} + \epsilon^{ln} \Gamma^{(km)} + \epsilon^{km} \Gamma^{(ln)} + \epsilon^{kn} \Gamma^{(lm)}), \quad (\text{B.20})$$

where Ω^{AB} denotes the $Sp(n)$ invariant tensor. In this paper we will only consider the fundamental spinor representations of $Sp(n)$ and $Sp(1)$, in which case

$$(\mathcal{D}_{Ai})_{Bn}{}^{B'n'} = \delta_B^{B'} \delta_n^{n'} \nabla_{Ai} + \delta_n^{n'} \omega_{Ai}{}^{B'} + \delta_B^{B'} \omega_{Ai}{}^{n'} . \quad (\text{B.21})$$

Particularly, we will be interested in the commutator of two covariant derivatives leading to the $Sp(n)$ and $Sp(1)$ components of the curvature tensor

$$[\mathcal{D}_{Ai}, \mathcal{D}_{Bj}]_{Cn}{}^{C'n'} = \delta_n^{n'} R_{Ai}{}_{Bj}{}^{C'} + \delta_C^{C'} R_{Ai}{}_{Bj}{}^{n'} =: R_{Ai, Bj}{}_{Cn}{}^{C'n'} . \quad (\text{B.22})$$

According to [17], restricting the holonomy group of \mathfrak{M} to (a subgroup of) $Sp(n) \times Sp(1)$ implies the covariant constraints⁹

$$R_{Ai}{}_{Bj}{}^{C'} = \epsilon_{ij} R_{AB;C}{}^{C'} , \quad \text{and} \quad R_{Ai}{}_{Bj}{}^{n'} = \Omega_{AB} R_{ij}{}^{n'} . \quad (\text{B.23})$$

Hence we have for the commutator

$$[\mathcal{D}_{Ai}, \mathcal{D}_{Bj}]_{Cn}{}^{C'n'} = -2 \delta_C^{C'} \Omega_{AB} R \Upsilon_{(ij)}{}^{n'} + \delta_n^{n'} \epsilon_{ij} [R_{(ABC}{}^{C')} - R (\Omega_{BC} \delta_A^{C'} + \Omega_{AC} \delta_B^{C'})] , \quad (\text{B.24})$$

where we have introduced

$$\Upsilon_{(ij)}{}^{n'} := \frac{1}{2} (\epsilon_{in} \delta_j^{n'} + \epsilon_{jn} \delta_i^{n'}) , \quad \text{and} \quad R := \frac{1}{6} R_{(ij)}{}^{(ij)} .$$

as the $Sp(1)$ generators analog to (B.3) and (B.4). Projecting with the corresponding harmonic variables, we particularly obtain

$$[\mathcal{D}_{A\pm}, \mathcal{D}_{B\pm}]_{Cn}{}^{C'n'} = \pm 2 \delta_C^{C'} \Omega_{AB} R (\Upsilon_{\pm\pm})_n{}^{n'} \quad (\text{B.25})$$

$$[\mathcal{D}_{A+}, \mathcal{D}_{B-}]_{Cn}{}^{C'n'} = -2 \delta_C^{C'} \Omega_{AB} R (\Upsilon_0)_n{}^{n'} + \delta_n^{n'} [R_{(ABC}{}^{C')} - R (\Omega_{BC} \delta_A^{C'} + \Omega_{AC} \delta_B^{C'})] , \quad (\text{B.26})$$

where we have introduced the projected quantities

$$\Upsilon_0 := \Upsilon_{+-} = \Upsilon_{-+} = u_+^i u_-^j \Upsilon_{ij}, \quad \text{and} \quad \Upsilon_{\pm\pm} = \mp \frac{1}{2} u_{\pm}^i u_{\pm}^j \Upsilon_{ij}. \quad (\text{B.27})$$

C Heterotic $K3 \times T^2$ Compactification

In this appendix we recall some important features of heterotic $K3 \times T^2$ compactifications, in particular expressions for various vertex operators and picture changing operators as well as our conventions for the spin-structure sum.

⁹The defining constraint of hyper-Kähler manifolds [21] implies that the right-hand side of (B.23) vanishes, while in the quaternionic case it becomes identical to the non-vanishing $Sp(1)$ part of the holonomy group.

C.1 Vertex Operators

The space-time and T^2 torus piece of all vertex operators is fairly standard and will be written using the free bosons $\phi_{1,2}$ and ϕ_3 respectively (for more information see e.g. [12]). For the $K3$ piece we recall that the $SU(2)$ current algebra inside the $\mathcal{N} = 4$ superconformal algebra can be bosonized in terms of a free boson which we call H :

$$J_{K3} = i\sqrt{2}\partial H, \quad J_{K3}^{\pm\pm} = e^{i\pm\sqrt{2}H}, \quad G_{K3,i}^{\pm} = e^{\pm\frac{i}{\sqrt{2}}H} \hat{G}_{K3,i}. \quad (C.1)$$

Here $\hat{G}_{K3,i}$ are dimension 5/4 operators which have non-singular OPE with H and have no spin structure dependence. The latter enters through the projections and shifts in the $U(1)$ charge lattice of J_{K3} which in turn is given by the momentum lattice of H . Therefore in the $\mathcal{N} = 4$ internal theory, only correlation functions and the partition function containing H depend on the spin-structure. The term in the picture changing operator containing a $\mathcal{N} = 4$ superconformal generator is

$$P = e^{\phi}(G_{K3,+}^{+} + G_{K3,-}^{-}) + \dots, \quad (C.2)$$

where dots indicate the remaining terms and ϕ bosonizes the superghost.

Concerning the vertex operators of physical fields, the chiral vector multiplet scalar in the (-1) -ghost picture is simply ψ_3 and in particular does not depend on the $K3$ fields. The gaugino vertex however involves, besides the space-time and torus spin-field, also $e^{\pm\frac{i}{\sqrt{2}}H}$. Thus the vertex operator for the gauginos λ_{\mp} and $\bar{\lambda}^{\pm}$ in the $(-1/2)$ -picture also carry $e^{\pm\frac{i}{\sqrt{2}}H}$.

Finally the vertex operators (at zero momentum) for the hyperscalar f_A^{\pm} in the (-1) and (0) ghost pictures are given by

$$V_{f_A^{\pm}}^{(-1)} = e^{-\phi} e^{\pm\frac{i}{\sqrt{2}}H} \hat{V}_A, \quad \text{and} \quad V_{f_A^{\pm}}^{(0)} = P V_{f_A^{\pm}}^{(-1)} =: \hat{G}_{K3,\pm} \hat{V}_A, \quad (C.3)$$

where \hat{V}_A have dimension 1/4 and have non-singular OPE with H . The normal ordered expression $:\hat{G}_{K3,\pm} \hat{V}_A:$ is defined as the coefficient of the $1/\sqrt{z}$ singularity in the corresponding OPE.

The vertex operators in the $(-1/2)$ ghost picture for the hyperfermions χ_A and $\bar{\psi}_A$ are

$$V_{\chi_A^{\alpha}}^{(-1/2)} = e^{-\frac{\phi}{2}} S^{\alpha} e^{-i\frac{\phi_3}{2}} \hat{V}_A, \quad \text{and} \quad V_{\bar{\psi}_A^{\dot{\alpha}}}^{(-1/2)} = e^{-\frac{\phi}{2}} S^{\dot{\alpha}} e^{i\frac{\phi_3}{2}} \hat{V}_A, \quad (C.4)$$

where S^{α} are the space-time spin fields.

As mentioned above the spin structure dependence enters only through the superghost, spacetime and torus fermions and the charge lattice of H . It does not depend in particular on the rest of the details of the $\mathcal{N} = 4$ superconformal theory. On the other hand, the topological theory (besides shifting the dimensions of the torus fermion) involves precisely twisting by adding an appropriate background charge for the field H and the rest of the internal $\mathcal{N} = 4$ theory is insensitive to this twisting.

C.2 Spin Structure

Let Γ be the $U(1)$ lattice of H charges. The space-time and torus fermions define an $SO(2) \times SO(2) \times SO(2)$ lattice. If one takes an $SO(2) \times SO(2)$ sublattice thereof and combines it with Γ , then (see [22, 23, 24]), the resulting 3-dimensional lattice is given by the coset $E_7/SO(8)$. The characters are given by the branching functions $F_{\Lambda,s}(\tau)$ and satisfy:

$$\chi_{\Lambda}(\tau) = \sum_s F_{\Lambda,s}(\tau) \chi_s(\tau), \quad (\text{C.5})$$

where χ_{Λ} and χ_s are the E_7 and $SO(8)$ level one characters respectively, Λ denotes the two conjugacy classes of E_7 and s represent the four conjugacy classes of $SO(8)$ in the spin structure basis. The characters of the internal $\mathcal{N} = 4$ superconformal field theory times two free complex fermions can therefore be expressed as $\sum_{\Lambda} F_{\Lambda,s}(\tau) Ch_{\Lambda}(\tau)$ where $Ch_{\Lambda}(\tau)$ is the contribution of the rest of the internal theory and most importantly does not depend on the spin-structure. The generalization to higher genus is obtained by assigning an E_7 conjugacy class Λ for each loop and we will denote this collection by $\{\Lambda\}$. We can define a more general character $F_{\{\Lambda\},s}(a_1, a_2, a_3)$ by introducing chemical potentials for the three charges; a_1 and a_2 coupling to the two $SO(2)$ charges and a_3 to H -charge. For a genus g surface, the couplings a_1 , a_2 and a_3 each are g -dimensional vectors and represent the coupling to charges going through each loop. In the calculation of the amplitudes (a_1, a_2, a_3) are related to the positions of various vertex operators weighted by the corresponding charges via Abel map. The spin structure sum is given by the formula:

$$\sum_s F_{\{\Lambda\},s}(a_1, a_2, a_3) = F_{\{\Lambda\}}\left(\frac{1}{2}(a_1 + a_2 + \sqrt{2}a_3), \frac{1}{2}(a_1 + a_2 - \sqrt{2}a_3), \frac{1}{\sqrt{2}}(a_1 - a_2)\right), \quad (\text{C.6})$$

where we have introduced

$$F_{\{\Lambda\}}(a_1, a_2, a_3) = \vartheta(\tau, a_1) \vartheta(\tau, a_2) \Theta(\tau, a_3), \quad (\text{C.7})$$

$$\Theta(\tau, a_3) = \sum_{n_i \in \mathbb{Z}} e^{2\pi i(n_i + \frac{\lambda_i}{2})\tau_{ij}(n_j + \frac{\lambda_j}{2}) + 2\pi i\sqrt{2}(n_i + \frac{\lambda_i}{2})(a_3)_i}, \quad (\text{C.8})$$

where λ_i (with $i = 1, \dots, g$) are 0 and 1 for the E_7 conjugacy classes **1** and **56** respectively. In fact, apart from the non-zero mode determinant of a scalar, Θ is just the character valued genus g partition function of level one $SU(2)$, with the two classes above corresponding to the two representations of level one $SU(2)$ Kac-Moody algebra based on $SU(2)$ representations **1** and **2**.

D Gauge Freedom

To see the gauge freedom of the potential for $\hat{\mathcal{H}}$ we consider the (α -dependent) potential function $\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\alpha)$ but insert the unit operator in the form of $\oint \bar{\psi}_3 \psi_3$ at a different position β . Then we deform the $\oint \bar{\psi}_3$ contour integral to obtain

$$(3g + n - 3)! \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\alpha) =$$

$$\begin{aligned}
&= \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-3} \psi_3(\alpha) \oint \bar{\psi}_3 \psi_3(\beta) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^{m_2} (\psi_3 \Xi^{B_c}) \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \\
&= \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-3} \oint \bar{\psi}_3 \psi_3(\alpha) \psi_3(\beta) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^{m_2} (\psi_3 \Xi^{B_c}) \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \\
&\quad + \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-3} \psi_3(\alpha) \psi_3(\beta) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \Xi^{[B_1]} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=2}^{m_2} (\psi_3 \Xi^{B_c])} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \\
&= (3g+n-3)! \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\beta) + (3g+n-3) \mathcal{D}_+^{[B_1]} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} \psi_3(\alpha) \psi_3(\beta) \cdot \\
&\quad \cdot \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^{m_2} (\psi_3 \Xi^{B_c])} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \tag{D.1}
\end{aligned}$$

This indeed shows that

$$\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\alpha) - \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2}}(\beta) = \mathcal{D}_+^{[B_1]} \mathcal{G}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2}}(\alpha, \beta), \tag{D.2}$$

with the quantity

$$\begin{aligned}
\mathcal{G}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2}}(\alpha, \beta) &:= \frac{1}{(3g+n-4)!} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} \psi_3(\alpha) \psi_3(\beta) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \cdot \\
&\quad \cdot \prod_{b=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=2}^{m_2} (\psi_3 \Xi^{B_c}) \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \tag{D.3}
\end{aligned}$$

For further convenience and in order to save writing, we will also introduce a generating functional for $\mathcal{G}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_2 \dots B_{m_2}}(\alpha, \beta)$

$$\begin{aligned}
\mathbb{G}^g(\alpha, \beta) &:= \sum_{n=0}^{\infty} \sum_{m_1=0}^n \sum_{m_2=0}^n \frac{\vartheta^{\bar{I}_1} \dots \vartheta^{\bar{I}_{n-m_1}} \chi_{A_1} \dots \chi_{A_{m_1+1}} \eta^{\bar{J}_1} \dots \eta^{\bar{J}_{n-m_2}} \xi_{B_1} \dots \xi_{B_{m_2-1}}}{(n-m_1)!(n-m_2)!(m_1+1)!(m_2+1)!} \cdot \\
&\quad \cdot \mathcal{G}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2-1}}(\alpha, \beta). \tag{D.4}
\end{aligned}$$

E Direct String Derivation of Differential Equations

E.1 String Derivation of Differential Equations

Our strategy in deriving differential equations with respect to $\mathcal{N} = 2$ moduli directly within the setup of topological string amplitudes follows closely [3]: Moduli derivatives are directly translated into vertex operator insertions within the twisted correlators. In particular we will consider derivatives with respect to vector multiplet and hypermultiplet moduli, for which the relevant insertions take the form

$$\mathcal{D}_{\bar{I}} \longleftrightarrow \int \oint G_{T^2}^+ \bar{\psi}_3 \bar{J}_{\bar{I}}, \quad \text{and} \quad \mathcal{D}_i^A \longleftrightarrow \int \oint G_{K3,i}^+ \bar{\Xi}^A. \tag{E.1}$$

We will then further deform the contour integrals (within the topologically twisted setup this is indeed possible on dimensional grounds) thereby trying to manipulate the correlation functions. A generic feature in this type of computations is the appearance of *world-sheet boundary terms*. The latter appear when the above mentioned contour integrals act on the measure of the twisted correlator, where they might give rise to insertions of the total energy momentum tensor $T = T_{T^2} + T_{K3}$ sewed with some Beltrami differential that parametrizes a particular deformation of the genus g -world-sheet. The full energy-momentum tensor can in fact be written as a total derivative with respect to this particular modulus of the genus g -Riemann surface, thus leading to a boundary contribution in the integral over \mathcal{M}_g . These boundary contributions can pictorially be quite directly figured as degeneration limits of the genus g Riemann surface and (see e.g. [7, 25, 26, 8, 11]) there are two homologically distinct types corresponding to the pinching of either a dividing geodesic or a handle. Examples for genus $g = 3$ are depicted in Figure 1 and 2 respectively. Assuming that we are at a generic

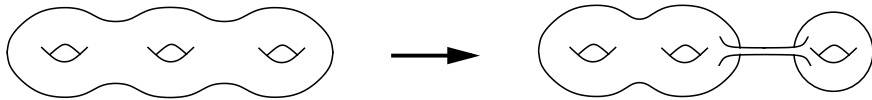


Figure 1: Pinching of a dividing geodesic of a genus $g = 3$ Riemann surface: The shrinking of a non-contractible cycle divides the Riemann surface into two new ones with genus $g = 2$ and $g = 1$ respectively.

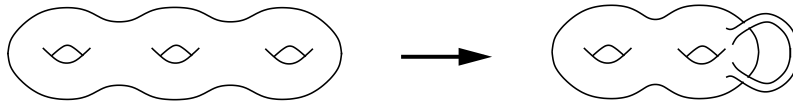


Figure 2: Pinching of a handle of a genus $g = 3$ Riemann surface: The shrinking of a non-contractible cycle results in a Riemann surface of genus $g = 2$ with two punctures.

point in the vector multiplet moduli-space with no charged massless states present there will be no contribution from pinching a handle. It therefore remains to study the pinching of a dividing geodesic where the genus g surface splits into two surfaces of genus g_s and $g - g_s$ respectively¹⁰ and to determine which states may propagate on the long and thin tube. In principle there are two different types, namely hyper- and vector multiplet states. We will discuss both in the following.

In our particular case, there is in fact a further type of degeneration besides these purely geometric ones. Indeed, since our world-sheets have in addition an arbitrary number of punctures (and the moduli space is thus rather $\mathcal{M}_{(g,n)}$) we also need to take into account contributions when two of these punctures collide with each other. Such contact terms are notoriously difficult to handle within the string approach. In our differential equations they

¹⁰Note that g_s can also take value zero or g since we have seen that these topological amplitudes start from genus zero.

will manifest themselves as curvature contributions and take into account the fact that the moduli space of the string compactification is in general not flat. We will not be able to say much about these terms in the general case, but we will propose consistent expressions for the case $m_2 = 0$ as we go along.

E.2 String Differential Equation for $\tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$

This appendix is devoted to the derivation of a differential equation for $\tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$ along the lines of section 4.1. The derivation will be divided into first considering the bulk contributions and afterwards the boundary terms, which appear from a particular deformation of the genus g world-sheet.

E.2.1 Bulk Equation

We start by explicitly applying the operator (4.2) to the potential $\tilde{\mathcal{H}}$ given in (2.29)

$$\epsilon^{ij} \frac{\partial}{\partial \bar{u}_+^i} \mathcal{D}_j^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1}], B_1 \dots B_{m_2+1}} = \frac{\epsilon^{ij}}{N} \frac{\partial}{\partial \bar{u}_+^i} \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-m_2-3} \cdot (\mu \cdot J_{K3}^{--}) \int \oint G_{K3,i}^+ \prod_{a=2}^{m_1} \int \Xi^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \quad (\text{E.2})$$

where for convenience we introduce the shorthand notations

$$N := (g + m_1 + m_2 - n)!(2g + 2n - m_1 - m_2 - 3)!, \quad (\text{E.3})$$

$$N' := (g + m_1 + m_2 - n)!(2g + 2n - m_1 - m_2 - 4)!, \quad (\text{E.4})$$

in order to save writing. Deforming the contour integral, we find two different terms

$$\begin{aligned} &= \frac{\epsilon^{ij}}{N'} \frac{\partial}{\partial \bar{u}_+^i} \left[\bar{u}_+^k \epsilon_{kj} \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-m_2-4} (\mu \cdot J_{K3}^{--}) (\mu \cdot T_{K3}) \cdot \right. \\ &\quad \cdot \left. \prod_{a=1}^{m_1} \int \Xi^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \right] + \\ &+ \frac{\epsilon^{ij}}{N} \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K3,+}^-)^{2g+2n-m_1-m_2-4} (\mu \cdot G_{K3,i}^-) (\mu \cdot G_{K3,j}^-) \prod_{a=1}^{m_1} \int \Xi^{A_a} \cdot \\ &\quad \cdot \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle. \end{aligned} \quad (\text{E.5})$$

The second term in fact vanishes due to the anti-symmetrization of the Beltrami differentials and the fermionic nature of the $G_{K3,i}^-$ operators. For the remaining term we can perform the

\bar{u}_+^i differential and furthermore use $T = T_{T^2} + T_{K^3}$. Denoting the boundary contribution as $\tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}$ we obtain

$$\begin{aligned}
&= -\frac{2g+2n-m_1-m_2-2}{N'} \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K^3,+}^-)^{2g+2n-m_1-m_2-4} (\mu \cdot J_{K^3}^-) \cdot \\
&\quad \cdot (\mu \cdot \oint G_{T^2}^+ G_{T^2}^-) \prod_{a=1}^{m_1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle + \\
&\quad + (2g+2n-m_1-m_2-2) \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}. \tag{E.6}
\end{aligned}$$

Deforming the contour integration, we obtain

$$\begin{aligned}
&= -\frac{2g+2n-m_1-m_2-2}{N'} \int_{\mathcal{M}_{(g,n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n+1} (\mu \cdot J_{K^3}^-) (\mu \cdot G_{K^3,+}^-)^{2g+2n-m_1-m_2-4} \cdot \\
&\quad \cdot \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1-1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \int \oint G_{T^2}^+ \bar{\psi}_3 \bar{J}_{\bar{I}_{n-m_1}} \prod_{d=1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle + \\
&\quad + (2g+2n-m_1-m_2-2) \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \tag{E.7}
\end{aligned}$$

The insertion $\int \oint G_{T^2}^+ \bar{\psi}_3 \bar{J}_{\bar{I}_{n-m_1}}$ is in fact a derivative with respect to $\mathcal{D}_{\bar{I}_{n-m_1}}$. Pulling it out and comparing with the definition (2.29) we find

$$\begin{aligned}
&(2g+2n-m_1-m_2-2) \mathcal{D}_{[\bar{I}_{n-m_1}} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} + \epsilon^{ij} \frac{\partial}{\partial \bar{u}_+^i} \mathcal{D}_j^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1}], B_1 \dots B_{m_2+1}} = \\
&= (2g+2n-m_1-m_2-2) \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}. \tag{E.8}
\end{aligned}$$

Notice that the prefactor is in fact the total harmonic charge of $\tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ shifted by 2, such that we can summarize this equation by writing

$$\begin{aligned}
&(-D_0+2) \mathcal{D}_{[\bar{I}_{n-m_1}} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1-1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} + \epsilon^{ij} \frac{\partial}{\partial \bar{u}_+^i} \mathcal{D}_j^{[A_1} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1}], B_1 \dots B_{m_2+1}} = \\
&= (-D_0+2) \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}}, \tag{E.9}
\end{aligned}$$

where we have explicitly pulled out the factor of $(2g+2n-m_1-m_2-2)$ from the boundary contribution. In the language of the generating functional, this latter factor is converted into the action of the operator $(-D_0+2)$ such that we can write the more compact relation

$$\left[(-D_0+2) \vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \epsilon^{ij} \frac{\partial}{\partial \bar{u}_+^i} \chi_A \mathcal{D}_j^A \right] \tilde{\mathbb{H}}^g = (-D_0+2) \tilde{\mathcal{C}}^{\text{bdy}}. \tag{E.10}$$

It still remains to determine the boundary contribution $\tilde{\mathcal{C}}^{\text{bdy}}$.

E.2.2 Boundary Contributions

The total contribution from the boundary deformations can be divided into two terms

$$\begin{aligned}
\tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} &= \int_{\mathcal{M}_{(g, n+1)}} \langle (\mu \cdot G_{T^2}^-)^{g+m_1+m_2-n} (\mu \cdot G_{K^3, +}^-)^{2g+2n-m_1-m_2-4} (\mu \cdot J_{K^3}^-) (\mu \cdot T) \cdot \\
&\quad \cdot \prod_{a=1}^{m_1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle = \\
&= \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}, \text{hyper}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} + \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}, \text{vector}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} \quad (\text{E.11})
\end{aligned}$$

corresponding to either hypermultiplet or vector multiplet states propagating on the long thin tube when pinching a dividing geodesic. We can calculate both contributions in a straight forward manner.

Hypermultiplet Propagation

There is not a unique way to distribute the operator insertions on the two Riemann surfaces but rather we have to consider several rather complicated contributions

$$\begin{aligned}
\tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}, \text{hyper}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} &= \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{(g_s, n_s)}} \langle (\mu \cdot G^-)^{3g_s+n_s-3} \prod_{a=1}^{m_1^s+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2^s} (\psi_3 \Xi^{[B_b]}) \cdot \\
&\quad \cdot \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n_s-m_2^s} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \oint G^- (\psi_3 \Xi^{B_1}) \rangle \cdot \Omega_{BC} \cdot \\
&\quad \cdot \int_{\mathcal{M}_{(g-g_s, n-n_s+1)}} \langle (\mu \cdot G^-)^{3(g-g_s)+n-n_s-3} \left[(\mu \cdot J_{K^3}^-) \oint (G^- \Xi^C) + (\mu \cdot G^-) \oint (J_{K^3}^- \Xi^C) \right] \prod_{a=m_1^s+2}^{m_1} \int \bar{\Xi}^{A_a} \cdot \\
&\quad \cdot \prod_{b=m_2^s+1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=n_s-m_2^s+1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle + \\
&+ \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{(g_s, n_s)}} \langle (\mu \cdot G^-)^{3g_s+n_s-4} \left[(\mu \cdot J_{K^3}^-) \oint G^- (\psi_3 \Xi^B) + (\mu \cdot G^-) \oint J_{K^3}^- (\psi_3 \Xi^B) \right] \cdot \\
&\quad \cdot \prod_{a=1}^{m_1^s} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2^s} (\psi_3 \Xi^{[B_b]}) \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n_s-m_2^s} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \rangle \cdot \Omega_{BC} \cdot \\
&\quad \cdot \int_{\mathcal{M}_{(g-g_s, n-n_s)}} \langle (\mu \cdot G^-)^{3(g-g_s)+n-n_s-2} \prod_{a=m_1^s+1}^{m_1} \int \bar{\Xi}^{A_a} \prod_{b=m_2^s+1}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \cdot \\
&\quad \cdot \prod_{d=n_s-m_2^s+1}^{n-m_2} (J_{K^3}^{++} \bar{J}_{\bar{J}_d}) \oint G^- \Xi^C \rangle \cdot \quad (\text{E.12})
\end{aligned}$$

Fortunately, this expression can be written in terms of topological amplitudes in the following manner

$$\begin{aligned}
& \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy,hyper}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} = \\
& = \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1^s+1}, \bar{J}_1 \dots \bar{J}_{n-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s} B} \Omega_{BC} \mathcal{D}_+^C \tilde{\mathcal{H}}_{\bar{I}_{n-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1}, B_{m_2^s+1} \dots B_{m_2+1}} + \\
& + \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \tilde{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m_1^s}, \bar{J}_1 \dots \bar{J}_{n-m_2^s}}^{g_s, A_1 \dots A_{m_1^s}, B_1 \dots B_{m_2^s} B} \Omega_{BC} \mathcal{D}_+^C \mathcal{H}_{\bar{I}_{n-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+1} \dots A_{m_1}, B_{m_2^s+1} \dots B_{m_2+1}} + \\
& + 2 \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1^s+1}, \bar{J}_1 \dots \bar{J}_{n-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s} B} \Omega_{BC} \mathcal{H}_{\bar{I}_{n-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1} C, B_{m_2^s+1} \dots B_{m_2+1}}. \quad (\text{E.13})
\end{aligned}$$

Vector Multiplet Propagation

Finally we also have to consider the possibility of having vector multiplet states propagate on the tubes. Also here we obtain several contributions

$$\begin{aligned}
& \tilde{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy,vector}), A_1 \dots A_{m_1}, B_1 \dots B_{m_2+1}} = \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{(g_s, n_s+1)}} \langle (\mu \cdot G^-)^{3g_s+n_s-2} \prod_{a=1}^{m_1^s+1} \int \Xi^{A_a} \prod_{b=1}^{m_2^s+1} (\psi_3 \Xi^{B_b}) \cdot \\
& \cdot \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n_s-m_2^s} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \oint G^- (\psi_3 \bar{J}_L) \rangle \cdot G^{L\bar{K}}. \\
& \cdot \int_{\mathcal{M}_{(g-g_s, n-n_s)}} \langle (\mu \cdot G^-)^{3(g-g_s)+n-n_s-4} \left[(\mu \cdot J_{K3}^{--}) \oint (G^- J_{K3}^{++} \bar{J}_{\bar{K}}) + (\mu \cdot G^-) \oint (J_{K3}^{--} J_{K3}^{++} \bar{J}_{\bar{K}}) \right] \cdot \\
& \cdot \prod_{a=m_1^s+2}^{m_1} \int \Xi^{A_a} \prod_{b=m_2^s+2}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=n_s-m_2^s+1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle + \\
& + \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{(g_s, n_s+1)}} \langle (\mu \cdot G^-)^{3g_s+n_s-3} \left[(\mu \cdot J_{K3}^{--}) \oint G^- (\psi_3 \bar{J}_L) + (\mu \cdot G^-) \oint J_{K3}^{--} (\psi_3 \bar{J}_L) \right] \cdot \\
& \cdot \prod_{a=1}^{m_1^s} \int \Xi^{A_a} \prod_{b=1}^{m_2^s+1} (\psi_3 \Xi^{B_b}) \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n_s-m_2^s} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \rangle \cdot G^{L\bar{K}}. \\
& \cdot \int_{\mathcal{M}_{(g-g_s, n-n_s)}} \langle (\mu \cdot G^-)^{3(g-g_s)+n-n_s-3} \prod_{a=m_1^s+1}^{m_1} \int \Xi^{A_a} \prod_{b=m_2^s+2}^{m_2+1} (\psi_3 \Xi^{B_b}) \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \cdot \\
& \cdot \prod_{d=n_s-m_2^s+1}^{n-m_2} (J_{K3}^{++} \bar{J}_{\bar{J}_d}) \oint G^- (J_{K3}^{++} \bar{J}_{\bar{K}}) \rangle. \quad (\text{E.14})
\end{aligned}$$

that can be fully expressed in terms of topological amplitudes

$$\begin{aligned}
& \tilde{\mathcal{C}}^{(\text{bdy,hyper}),A_1\dots A_{m_1},B_1\dots B_{m_2+1}}_{\bar{I}_1\dots\bar{I}_{n-m_1+1},\bar{J}_1\dots\bar{J}_{n-m_2}} = \\
& = \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \partial_L \mathcal{H}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s+1},B_1\dots B_{m_2^s}B} G^{L\bar{K}} \tilde{\mathcal{H}}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+2}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} + \\
& + \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \partial_L \tilde{\mathcal{H}}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s},B_1\dots B_{m_2^s}B} G^{L\bar{K}} \mathcal{H}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+1}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} \bar{K}.
\end{aligned} \tag{E.15}$$

The final result of the boundary is given by equation (E.11) with the expressions (E.13) and (E.15), i.e.

$$\begin{aligned}
& \tilde{\mathcal{C}}^{(\text{bdy}),A_1\dots A_{m_1},B_1\dots B_{m_2+1}}_{\bar{I}_1\dots\bar{I}_{n-m_1+1},\bar{J}_1\dots\bar{J}_{n-m_2}} = \\
& = \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s+1},B_1\dots B_{m_2^s}B} \Omega_{BC} \mathcal{D}_+^C \tilde{\mathcal{H}}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+2}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} + \\
& + \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \tilde{\mathcal{H}}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s},B_1\dots B_{m_2^s}B} \Omega_{BC} \mathcal{D}_+^C \mathcal{H}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+1}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} + \\
& + 2 \sum_{g_s=1}^{g-1} \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s+1},B_1\dots B_{m_2^s}B} \Omega_{BC} \mathcal{H}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+2}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} + \\
& + \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} D_L \mathcal{H}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s+1},B_1\dots B_{m_2^s}B} G^{L\bar{K}} \tilde{\mathcal{H}}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+2}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} \bar{K} + \\
& + \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} D_L \tilde{\mathcal{H}}_{\bar{I}_1\dots\bar{I}_{n_s-m_1^s},\bar{J}_1\dots\bar{J}_{n_s-m_2^s}}^{g_s,A_1\dots A_{m_1^s},B_1\dots B_{m_2^s}B} G^{L\bar{K}} \mathcal{H}_{\bar{I}_{n_s-m_1^s+1}\dots\bar{I}_{n-m_1+1},\bar{J}_{n_s-m_2^s+1}\dots\bar{J}_{n-m_2}}^{g-g_s,A_{m_1^s+1}\dots A_{m_1},B_{m_2^s+1}\dots B_{m_2+1}} \bar{K}.
\end{aligned} \tag{E.16}$$

This expression is more compactly written using the generating functional (2.22)

$$\begin{aligned}
& \tilde{\mathbb{C}}^{(\text{bdy})} = \sum_{g_s=0}^g \left[\left(\mathcal{D}_+^A \tilde{\mathbb{H}}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) - \left(\mathcal{D}_+^A \mathbb{H}^{g_s} \right) \Omega_{AB} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \xi_B} \right) + \right. \\
& \left. + \left(\frac{\partial \mathbb{H}^{g_s}}{\partial \chi_A} \right) \Omega_{AB} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \xi_B} \right) + \left(\mathcal{D}_I \tilde{\mathbb{H}}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) - \left(\mathcal{D}_I \mathbb{H}^{g_s} \right) G^{I\bar{J}} \left(\frac{\partial \tilde{\mathbb{H}}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right].
\end{aligned} \tag{E.17}$$

E.3 String Differential Equation for $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$

E.3.1 Bulk Equation

An alternative (and slightly more technical) method to arrive at (4.13) is to work directly with the string expression $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$. To this end we start by taking a derivative with respect to an anti-holomorphic vector multiplet scalar

$$\begin{aligned} \mathcal{D}_{[\bar{I}_{n-m_1+1}} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}], \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \frac{1}{(3g-2+n)!} \cdot \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-2+n} \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \\ &\cdot \prod_{c=1}^{n-m_1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \oint G^+ \bar{\psi}_3 \bar{J}_{\bar{I}_{n-m_1+1}} \prod_{b=1}^{m_2+1} \psi_3 \Xi^{B_b} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle, \end{aligned}$$

where we have introduced the operator

$$G^+ = G_{T^2}^+ + G_{K3,-}^+. \quad (\text{E.18})$$

Notice that similar to G^- also this operator is strictly speaking not well defined regarding its harmonic charge. However, just as before the necessity of soaking up all torus fermionic zero modes ψ_3 takes care of this ambiguity in a well defined way. Deforming now the $\oint G^+$ -contour we obtain

$$\begin{aligned} \mathcal{D}_{[\bar{I}_{n-m_1+1}} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}], \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} + \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bulk}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \\ &= (-1)^{n+1} \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-3+n} (\mu \cdot T) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{c=1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{b=1}^{m_2+1} \psi_3 \Xi^{B_b} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle + \\ &+ (-1)^n \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-2+n} \oint G_{K3,-}^+ \bar{\Xi}^{[A_1} \prod_{a=2}^{m_1+1} \int \bar{\Xi}^{A_a]} \prod_{c=1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{b=1}^{m_2+1} \psi_3 \Xi^{B_b} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle \end{aligned}$$

The first term $\mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ corresponds to a boundary contribution since it contains an insertion of the full energy-momentum tensor. The term $\mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bulk}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$ on the other hand is a bulk contribution and is in fact just a vector multiplet derivative of $\mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}$. Thus we can write

$$\begin{aligned} \mathcal{D}_{[\bar{I}_{n-m_1+1}} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}], \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &+ (-1)^{n+1} \mathcal{D}_{-}^{[A_1} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_2 \dots A_{m_1+1}], B_1 \dots B_{m_2+1}} \\ &= (-1)^{n+1} \int_{\mathcal{M}_g} \langle (\mu \cdot G^-)^{3g-3+n} (\mu \cdot T) \prod_{a=1}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{c=1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{b=1}^{m_2+1} \psi_3 \Xi^{B_b} \prod_{d=1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \rangle \\ &= \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}}. \end{aligned} \quad (\text{E.19})$$

E.3.2 Boundary Contribution

We can directly discuss the contribution of vector- and hypermultiplets separately.

Hypermultiplet Propagation

Assuming that the two surfaces have genus g_s and $g - g_s$ respectively we get the contribution

$$\begin{aligned}
& \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy,hyper}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \\
& = \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{g_s}} \langle (\mu \cdot G^-)^{3g_s-3+n_s} \prod_{a=1}^{m_1^s+1} \int \bar{\Xi}^{A_a} \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{b=1}^{m_2^s} \psi_3 \Xi^{B_b} \prod_{d=1}^{n_s-m_2^s} J_{K3}^{++} \bar{J}_{\bar{J}_d} \cdot \\
& \cdot \oint G^- (\psi_3 \Xi^{B_b}) \rangle \Omega_{BC} \int_{\mathcal{M}_{g-g_s}} \langle (\mu \cdot G^-)^{3(g-g_s)-2+(n-n_s)} \prod_{a=m_1^s+2}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \cdot \\
& \cdot \prod_{b=m_2^s+1}^{m_2+1} \psi_3 \Xi^{B_b} \prod_{d=n_s-m_2^s+1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \oint G^- \Xi^C \rangle. \tag{E.20}
\end{aligned}$$

Here the operator $\psi_3 \Xi^B$ on genus g_s and the operator Ξ^C on genus $g - g_s$ surfaces appear at the nodes respectively and Ω_{BC} is the propagator on the tube. These operators are dimension zero and the net charge is $+3$ as it should be for the sphere. In the above expression G^- contours around these operators are coming from the Beltrami differentials associated with the punctures at the nodes. Note however that when g_s or $g - g_s$ is one, then the operator at the node on the genus 1 surface is unintegrated and the corresponding G^- is folded with the Beltrami differential dual to the torus modulus. Similarly when one of the surfaces is of genus zero then three of the dimension zero operators on the sphere are unintegrated. As a result the counting of G^- in all cases gives the correct values.

Reinterpreting this expression making use of the anti-symmetrization of the B -indices we obtain the expression

$$\begin{aligned}
& \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy,hyper}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \\
& = \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1^s}, \bar{J}_1 \dots \bar{J}_{n-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s} B} \Omega_{BC} \mathcal{D}_+^C \mathcal{H}_{\bar{I}_{n_s-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n_s-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1+1}, B_{m_2^s+1} \dots B_{m_2+1}}. \tag{E.21}
\end{aligned}$$

Vector Multiplet Propagation

In the case $m_1 \neq 0 \neq m_2$ we will also obtain vector multiplet contributions for the pinching of Riemann surfaces of generic genus.

$$\begin{aligned}
& \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(\text{bdy,vector}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} = \\
& = \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \int_{\mathcal{M}_{g_s}} \langle (\mu \cdot G^-)^{3g_s-2+n_s} \prod_{a=1}^{m_1^s+1} \int \bar{\Xi}^{A_a} \prod_{b=1}^{m_2^s+1} \psi_3 \Xi^{B_b} \prod_{c=1}^{n_s-m_1^s} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=1}^{n_s-m_2^s} J_{K3}^{++} \bar{J}_{\bar{J}_d} \cdot \\
& \cdot \oint G^- (\psi_3 \bar{J}_{\bar{L}}) \rangle G^{L\bar{K}} \int_{\mathcal{M}_{g-g_s}} \langle (\mu \cdot G^-)^{3(g-g_s)-3+(n-n_s)} \prod_{a=m_1^s+2}^{m_1+1} \int \bar{\Xi}^{A_a} \prod_{b=m_2^s+2}^{m_2+1} \psi_3 \Xi^{B_b} \cdot
\end{aligned}$$

$$\cdot \prod_{c=n_s-m_1^s+1}^{n-m_1+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_c} \prod_{d=n_s-m_2^s+1}^{n-m_2} J_{K3}^{++} \bar{J}_{\bar{J}_d} \oint G^- (J_{K3}^{++} \bar{J}_{\bar{K}}) \rangle. \quad (\text{E.22})$$

Notice that for $m_2 = 0$ there will only be contributions to this expression if one of the two surfaces is genus 1 or 0, otherwise there will not be enough ψ_3 -insertions on both Riemann surfaces to soak up all zero modes (see also section 5.1). Rewriting this expression using once more the anti-symmetrization of the B -indices, we obtain the following relation

$$\begin{aligned} \mathcal{C}_{\bar{I}_1 \dots \bar{I}_{n-m_1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{(1, \text{vector}), A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} &= \\ &= \mathcal{D}_L \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n_s-m_1^s}, \bar{J}_1 \dots \bar{J}_{n_s-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s+1}} G^{L\bar{K}} \mathcal{H}_{\bar{I}_{n_s-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n_s-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1+1}, B_{m_2^s+2} \dots B_{m_2+1}} \bar{K}. \end{aligned} \quad (\text{E.23})$$

Taking all results together we find for the equation (E.19)

$$\begin{aligned} \mathcal{D}_{[\bar{I}_{n-m_1+1}} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1}], \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1+1}, B_1 \dots B_{m_2+1}} + (-1)^{n-m_1+1} \mathcal{D}_{-}^{[A_{m_1+1}] \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m_1+1}, \bar{J}_1 \dots \bar{J}_{n-m_2}}^{g, A_1 \dots A_{m_1}], B_1 \dots B_{m_2+1}} &= \\ = \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n_s-m_1^s}, \bar{J}_1 \dots \bar{J}_{n_s-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s}} \Omega_{BC} \mathcal{D}_{+}^C \mathcal{H}_{\bar{I}_{n_s-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n_s-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1+1}, B_{m_2^s+2} \dots B_{m_2+1}} &+ \\ + \sum_{g_s=0}^g \sum_{n_s=0}^n \sum_{m_1^s=0}^{n_s} \sum_{m_2^s=0}^{n_s} \mathcal{D}_L \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n_s-m_1^s}, \bar{J}_1 \dots \bar{J}_{n_s-m_2^s}}^{g_s, A_1 \dots A_{m_1^s+1}, B_1 \dots B_{m_2^s+1}} G^{L\bar{K}} \mathcal{H}_{\bar{I}_{n_s-m_1^s+1} \dots \bar{I}_{n-m_1+1}, \bar{J}_{n_s-m_2^s+1} \dots \bar{J}_{n-m_2}}^{g-g_s, A_{m_1^s+2} \dots A_{m_1+1}, B_{m_2^s+2} \dots B_{m_2+1}} \bar{K}. \end{aligned} \quad (\text{E.24})$$

Since this equation is very involved due to the many indices of the various expressions, we resort to reformulating it in terms of the generating functional (2.22)

$$\left(\vartheta^{\bar{I}} \mathcal{D}_{\bar{I}} + \chi_A \mathcal{D}_{-}^A \right) \mathbb{H}^g = \sum_{g_s=0}^g \left[\left(\frac{\partial \mathbb{H}^{g_s}}{\partial \xi_A} \right) \Omega_{AB} (\mathcal{D}_{+}^B \mathbb{H}^{g-g_s}) + (\mathcal{D}_I \mathbb{H}^{g_s}) G^{I\bar{J}} \left(\frac{\partial \mathbb{H}^{g-g_s}}{\partial \eta^{\bar{J}}} \right) \right]. \quad (\text{E.25})$$

Up to possible contact terms (which are incarnations of the curvature contributions), which we did not consider in the computation so far, this precisely agrees with (4.13). The left hand side of this equation can be attributed as a bulk equation, while the right hand side is an additional boundary contribution. Since in the present case only boundaries corresponding to degeneration of the genus g surface along dividing geodesics contribute, these anomalous terms should be due to reducible graphs involving elimination of auxiliary fields.

E.4 String Differential Equation for $\hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_m}$

We start by considering a differentiation of (2.44) with respect to one of the vector multiplet moduli

$$\begin{aligned} \mathcal{D}_{[\bar{I}_{m-n+1}} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m}], \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_m} &= \frac{1}{(3g+n-4)!} \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} (\mu \cdot J_{K3}^-) \psi_3(\alpha) \cdot \\ &\cdot \prod_{a=1}^m \int \Xi^{A_a} \prod_{b=1}^{n-m} \int \bar{\psi}_3 \bar{J}_{[\bar{I}_b} \prod_{c=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_c}) \oint G^+ \bar{\psi}_3 \bar{J}_{\bar{I}_{m-n+1}]} \rangle. \end{aligned}$$

Deforming the contour integral of the G^+ operator we obtain the following three contributions

$$\begin{aligned}
\mathcal{D}_{[\bar{I}_{m-n+1}]} \mathcal{H}_{\bar{I}_1 \dots \bar{I}_{n-m}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_m} &= \hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy}), A_1 \dots A_m} + \hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bulk}), A_1 \dots A_m} = \\
&= (-1)^n \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-5} (\mu \cdot T) (\mu \cdot J_{K3}^-) \psi_3(\alpha) \prod_{a=1}^m \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_c}) \rangle \\
&+ (-1)^n \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} (\mu \cdot G_{K3,-}^-) \psi_3(\alpha) \prod_{a=1}^m \int \bar{\Xi}^{A_a} \prod_{b=1}^{n-m+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^n (J_{K3}^{++} \bar{J}_{\bar{J}_c}) \rangle \\
&+ (-1)^n \int_{\mathcal{M}_{(g,n)}} \langle (\mu \cdot G^-)^{3g+n-4} (\mu \cdot J_{K3}^{--}) \psi_3(\alpha) \oint G_{K3,-}^+ \bar{\Xi}^{[A_1} \prod_{a=2}^m \int \bar{\Xi}^{A_a]} \prod_{b=1}^{n-m+1} \int \bar{\psi}_3 \bar{J}_{\bar{I}_b} \prod_{c=1}^n J_{K3}^{++} \bar{J}_{\bar{J}_c} \rangle
\end{aligned} \tag{E.26}$$

The first line $\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy}), A_1 \dots A_m}$ is a boundary contribution due to the insertion of the energy momentum tensor sewed with one of the Beltrami differentials. Following the same lines as before, we can immediately state the result

$$\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy}), A_1 \dots A_m} = \hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy,hyper}), A_1 \dots A_m} + \hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy,vector}), A_1 \dots A_m}, \tag{E.27}$$

with the explicit expressions

$$\begin{aligned}
\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy,hyper}), A_1 \dots A_m} &= \sum_{g_1=0}^g \sum_{n_s=0}^n \sum_{m_s=0}^{n_s} \mathcal{D}_+^B \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n_s-m_s}, \bar{J}_1 \dots \bar{J}_{n_s}}^{g, A_1 \dots A_{m_s}} \Omega_{BC} \mathcal{D}_+^C \hat{\mathcal{H}}_{\bar{I}_{n_s-m_s+1} \dots \bar{I}_{n-m+1}, \bar{J}_{n_s+1} \dots \bar{J}_n}^{g, A_{m_s+1} \dots A_m}, \\
\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bdy,vector}), A_1 \dots A_m} &= 0.
\end{aligned}$$

Here the propagation of the vector multiplet states is in fact cancelled against the contribution stemming from the exchange of a state ρ of charge $+3$ and the unit operator. As already remarked in footnote ⁶, this subtle cancellation was overlooked in the equations derived in [12], but is strictly necessary for the integrability of the final differential equation. For later convenience, we introduce a generating functional for the quantity (E.27)

$$\hat{\mathbb{C}}_{\text{bdy}}^g := \sum_{g_s=1}^{g-1} \mathcal{D}_+^B \hat{\mathbb{H}}_{\text{red}}^g \Omega_{BC} \mathcal{D}_+^C \hat{\mathbb{H}}_{\text{red}}^g. \tag{E.28}$$

In the remaining two lines of (E.26) which we called $\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bulk}), A_1 \dots A_m}$ we readily recognize again a bulk contribution, which we may rewrite using the harmonicity operator (4.2)

$$\hat{\mathcal{C}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{(\text{bulk}), A_1 \dots A_m} = (-1)^{n-m_1} \nabla_-^{[A_m} \hat{\mathcal{H}}_{\bar{I}_1 \dots \bar{I}_{n-m+1}, \bar{J}_1 \dots \bar{J}_n}^{g, A_1 \dots A_{m-1}]} \tag{E.29}$$

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